m3a22soln8.tex

SOLUTIONS 8. 11.12.2015

Q1.

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)dx = \int_{-\infty}^{\infty} e^{tx}.\frac{1}{\sqrt{2\pi}\sigma} \exp\{-\frac{1}{2}(x-\mu)^2/\sigma^2\}dx.$$

Make the substitution $u := (x-\mu)/\sigma$: $x = \mu + \sigma u$, $dx = \sigma du$:

$$M_X(t) = \int_{-\infty}^{\infty} e^{t(\mu+\sigma u)}.\frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}u^2\}du = e^{\mu t}.\int_{-\infty}^{\infty} e^{\sigma tu}.\frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}u^2\}du.$$

Completing the square in the exponent on the right,

$$M(t) = e^{\mu t}.\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}[u^2 - 2\sigma tu]\}du$$

$$= e^{\mu t}.\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}[(u-\sigma t)^2 - \sigma^2 t^2]\}du = e^{\mu t + \frac{1}{2}\sigma^2 t^2}.\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}(u-\sigma t)^2\}du.$$

The integral on the right is 1 (a density integrates to 1 – of $N(\sigma t, 1)$ as it stands, or of $N(0,1)$ after the substitution $v := u - \sigma t$), giving

$$M(t) = \exp\{\mu t + \frac{1}{2}\sigma^2 t^2\}.$$

Q2. (i) By Q1,

$$M_Y(t) = E[e^{tY}] = \exp\{\mu t + \frac{1}{2}\sigma^2 t^2\}.$$

Taking $t = 1$,

$$M_Y(1) = E[e^Y] = \exp\{\mu + \frac{1}{2}\sigma^2\}.$$

As $X = e^Y$, this gives

$$E[X] = E[e^Y] = e^{\mu + \frac{1}{2}\sigma^2}.$$

(ii) In the Black-Scholes model, stock prices are geometric Brownian motions, driven by stochastic differential equations

$$dS = S(\mu dt + \sigma dB), \qquad (GBM)$$

with B Brownian motion. This has solution (we quote this – from Itô's lemma – Ch. V W9)

$$S_t = S_0 \exp\{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t\}.$$

So $\log S_t = \log S_0 + (\mu - \frac{1}{2}\sigma^2)t + \sigma B_t$ is normally distributed, so S_t is lognormal.

NHB

Q3. In Q1, t is real, but if we formally replace t by it, we get the normal CF as

$$E[e^{itX}] = \exp\{i\mu t - \frac{1}{2}\sigma^2 t^2\}.$$

This is indeed correct, but a formal proof needs some Complex Analysis. There are two ways to see this:

(i) *Analytic continuation.* If we let t in Q1 be *complex*, the MGF $\exp\{\mu t + \frac{1}{2}\sigma^2 t^2\}$ becomes an analytic (= holomorphic) function with no singularities in the whole complex t-plane $\mathbb{C}$ – that is, an *entire* (= integral) function. For entire functions, 'what looks right, is right', by *analytic continuation* (similarly for analytic functions, within domains of analyticity). For background, see any decent book on Complex Analysis, or e.g. my home-page, M2P3 Complex Analysis link, 2011 L 22 - 23. The technique is very powerful, and well worth mastering.

(ii) *Cauchy's (Residue) Theorem.* Alternatively, one can prove this by integrating the function $e^{\frac{1}{2}z^2}$ round a long thin rectangle in the complex z-plane, and using Cauchy's (Residue) Theorem (actually, there are no residues, as there are no singularities – as above). See e.g. M2P3 L 26 - 27.