

SOLUTIONS 7. 4.12.2015

Q1. *Vega for calls.* With $\phi(x) := e^{-\frac{1}{2}x^2}/\sqrt{2\pi}$, $\Phi(x) := \int_{-\infty}^x \phi(u)du$ the standard normal density and distribution functions, $\tau := T - t$ the time to expiry, the Black-Scholes call price is

$$C_t := S_t \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2), \quad (BS)$$

$$d_1 := \frac{\log(S/K) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}, \quad d_2 := \frac{\log(S/K) + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} = d_1 - \sigma\sqrt{\tau} :$$

$$\phi(d_2) = \phi(d_1 - \sigma\sqrt{\tau}) = \frac{e^{-\frac{1}{2}(d_1 - \sigma\sqrt{\tau})^2}}{\sqrt{2\pi}} = \frac{e^{-\frac{1}{2}d_1^2}}{\sqrt{2\pi}} \cdot e^{d_1\sigma\sqrt{\tau}} \cdot e^{-\frac{1}{2}\sigma^2\tau} :$$

$$\phi(d_2) = \phi(d_1) \cdot e^{d_1\sigma\sqrt{\tau}} \cdot e^{-\frac{1}{2}\sigma^2\tau}.$$

Exponentiating the definition of d_1 ,

$$e^{d_1\sigma\sqrt{\tau}} = (S/K) \cdot e^{r\tau} \cdot e^{\frac{1}{2}\sigma^2\tau}.$$

Combining,

$$\phi(d_2) = \phi(d_1) \cdot (S/K) \cdot e^{r\tau} : \quad K e^{-r\tau} \phi(d_2) = S \phi(d_1). \quad (*)$$

Differentiating (BS) partially w.r.t. σ gives

$$v := \partial C / \partial \sigma = S \phi(d_1) \partial d_1 / \partial \sigma - K e^{-r\tau} \phi(d_2) \partial d_2 / \partial \sigma.$$

So by (*),

$$v := \partial C / \partial \sigma = S \phi(d_1) \partial (d_1 - d_2) / \partial \sigma = S \phi(d_1) \partial (\sigma\sqrt{\tau}) / \partial \sigma = S \phi(d_1) \sqrt{\tau} > 0.$$

Vega for puts.

The same argument gives $v := \partial P / \partial \sigma > 0$, starting with the Black-Scholes formula for puts. Equivalently, we can use put-call parity

$$S + P - C = K e^{-r\tau} : \quad \partial P / \partial \sigma = \partial C / \partial \sigma > 0.$$

Interpretation: "Options like volatility": the more uncertainty, i.e. the higher the volatility, the more the "insurance policy" of an option is worth. So vega

is positive for positions *long* in the option – but negative for *short* positions.

Q2.(i) *Delta for calls.*

$$\begin{aligned}\Delta &:= \partial C / \partial S = \frac{\partial}{\partial S} [S\Phi(d_1) - Ke^{-r\tau}\Phi(d_2)] \\ &= \Phi(d_1) + S\phi(d_1)\frac{\partial d_1}{\partial S} - Ke^{-r\tau}\phi(d_2)\frac{\partial d_2}{\partial S} \\ &= \Phi(d_1) + S\phi(d_1)\frac{\partial(d_1 - d_2)}{\partial S},\end{aligned}$$

by Q1 (*). Since $d_1 - d_2 = \sigma\sqrt{\tau}$ does not depend on S , this gives

$$\Delta = \Phi(d_1) \in (0, 1).$$

Interpretation: the payoff $(S - K)_+$ is increasing in S , so the option price should be also – and it is: $\Delta > 0$.

Also, $\Delta < 1$: options are to insure against adverse price movements. This reflects that options are useful for this: if Δ were ≥ 1 , there would be no advantage in using options to hedge – we would just use a combination of cash and stock.

(ii) *Delta for puts.* Now put-call parity

$$S + P - C = Ke^{-r\tau}$$

and (i) give

$$\partial P / \partial S = \partial C / \partial S - 1 \in (-1, 0).$$

Interpretation: now the payoff $(K - S)_+$ is decreasing in S , so the option price should be also – and it is. That $\Delta > -1$ reflects that options are useful for insuring against adverse price movements (as above): if Δ were ≤ -1 , we would just use a combination of cash and stock.

Q3. *Vega for American options.*

The discounted value of an American option is the Snell envelope $\tilde{U}_{n-1} = \max(\tilde{Z}_{n-1}, E^*[\tilde{U}_n | \mathcal{F}_{n-1}])$ of the discounted payoff \tilde{Z}_n (exercised early at time $n < N$), with terminal condition $U_N = Z_N, \tilde{U}_N = \tilde{Z}_N$. As σ increases, the Z s increase (Q1: vega is positive for European options). As the Z s increase, the U s increase (above: backward induction on n – DP, as usual for American options). Combining: as σ increases, the U s increase. So vega is also positive for American options. //

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