m3a22soln4.tex

SOLUTIONS 4. 13.11.2015

Q1. Since f is clearly non-negative, to show that f is a (probability density) function (in two dimensions), it suffices to show that f integrates to 1:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1, \qquad \text{or} \qquad \int\int f = 1.$$

Write

$$f_1(x) := \int_{-\infty}^{\infty} f(x,y)dy, \qquad f_2(y) := \int_{-\infty}^{\infty} f(x,y)dx.$$

Then to show $\int\int f = 1$, we need to show $\int_{-\infty}^{\infty} f_1(x)dx = 1$ (or $\int_{-\infty}^{\infty} f_2(y)dy = 1$). Then f_1, f_2 are densities, in *one* dimension. If $f(x,y) = f_{X,Y}(x,y)$ is the *joint* density of *two* random variables X, Y, then $f_1(x)$ is the density $f_X(x)$ of X, $f_2(y)$ the density $f_Y(y)$ of Y (f_1, f_2, or f_X, f_Y, are called the *marginal* densities of the *joint* density f, or $f_{X,Y}$).

To perform the integrations, we have to *complete the square.* We have the algebraic identity

$$(1-\rho^2)Q \equiv \Big[\Big(\frac{y-\mu_2}{\sigma_2}\Big) - \rho\Big(\frac{x-\mu_1}{\sigma_1}\Big)\Big]^2 + (1-\rho^2)\Big(\frac{x-\mu_1}{\sigma_1}\Big)^2$$

(reducing the number of occurrences of y to 1, as we intend to integrate out y first). Then (taking the terms free of y out through the y-integral)

$$f_1(x) = \frac{\exp(-\frac{1}{2}(x-\mu_1)^2/\sigma_1^2)}{\sigma_1\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sigma_2\sqrt{2\pi}\sqrt{1-\rho^2}} \exp\left(\frac{-\frac{1}{2}(y-c_x)^2}{\sigma_2^2(1-\rho^2)}\right) dy, \qquad (*)$$

where

$$c_x := \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1).$$

The integral is 1 ('normal density'). So

$$f_1(x) = \frac{\exp(-\frac{1}{2}(x-\mu_1)^2/\sigma_1^2)}{\sigma_1\sqrt{2\pi}},$$

which integrates to 1 ('normal density'), proving

Fact 1. $f(x,y)$ is a joint density function (two-dimensional), with marginal density functions $f_1(x), f_2(y)$ (one-dimensional). So we can write

$$f(x,y) = f_{X,Y}(x,y), \qquad f_1(x) = f_X(x), \qquad f_2(y) = f_Y(y).$$

Fact 2. X, Y are normal: X is $N(\mu_1, \sigma_1^2)$, Y is $N(\mu_2, \sigma_2^2)$. For, we showed $f_1 = f_X$ to be the $N(\mu_1, \sigma_1^2)$ density above, and similarly for Y by symmetry.
Fact 3. $EX = \mu_1, EY = \mu_2, varX = \sigma_1^2, varY = \sigma_2^2$.

This identifies four out of the five parameters: two means μ_i, two variances σ_i^2. Next, recall conditional densities [L9]:

$$f_{Y|X}(y|x) := f_{X,Y}(x,y)/f_X(x) = f_{X,Y}(x,y)/\int_{-\infty}^{\infty} f_{X,Y}(x,y)dy.$$

Returning to the bivariate normal:
Fact 4. The conditional distribution of y given $X = x$ is $N(\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x - \mu_1), \quad \sigma_2^2(1-\rho^2))$.
Proof. Go back to completing the square (or, return to (*) with $\int$ and dy deleted):

$$f(x,y) = \frac{\exp(-\frac{1}{2}(x-\mu_1)^2/\sigma_1^2)}{\sigma_1\sqrt{2\pi}} . \frac{\exp(-\frac{1}{2}(y-c_x)^2/(\sigma_2^2(1-\rho^2)))}{\sigma_2\sqrt{2\pi}\sqrt{1-\rho^2}}.$$

The first factor is $f_1(x)$, by Fact 1. So, $f_{Y|X}(y|x) = f(x,y)/f_1(x)$ is the second factor:

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}\sigma_2\sqrt{1-\rho^2}} \exp\Big(\frac{-(y-c_x)^2}{2\sigma_2^2(1-\rho^2)}\Big),$$

where c_x is the linear function of x given below (*). //

This not only completes the proof of Fact 4 but gives Facts 5 and 6:
Fact 5. The conditional mean $E(Y|X = x)$ is *linear* in x:

$$E(Y|X = x) = \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x - \mu_1).$$

Fact 6. The conditional variance of Y given $X = x$ is

$$var(Y|X = x) = \sigma_2^2(1-\rho^2).$$

Fact 7. The correlation coefficient of X, Y is ρ.
Proof.

$$\rho(X,Y) := E\Big[\Big(\frac{X-\mu_1}{\sigma_1}\Big)\Big(\frac{Y-\mu_2}{\sigma_2}\Big)\Big] = \int\int\Big(\frac{x-\mu_1}{\sigma_1}\Big)\Big(\frac{y-\mu_2}{\sigma_2}\Big)f(x,y)dxdy.$$

Substitute for $f(x,y) = c\exp(-\frac{1}{2}Q)$, and make the change of variables $u := (x-\mu_1)/\sigma_1$, $v := (y-\mu_2)/\sigma_2$:

$$\rho(X,Y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int\int uv \exp\Big(\frac{-[u^2 - 2\rho uv + v^2]}{2(1-\rho^2)}\Big) du dv.$$

Completing the square as before, $[u^2 - 2\rho uv + v^2] = (v-\rho u)^2 + (1-\rho^2)u^2$. So

$$\rho(X,Y) = \frac{1}{\sqrt{2\pi}} \int u \exp\left(-\frac{u^2}{2}\right) du . \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} \int v \exp\left(-\frac{(v-\rho u)^2}{2(1-\rho^2)}\right) dv.$$

Replace v in the inner integral by $(v-\rho u)+\rho u$, and calculate the two resulting integrals separately. The first is zero ('normal mean', or symmetry), the second is ρu ('normal density'). So

$$\rho(X,Y) = \frac{1}{\sqrt{2\pi}} . \rho \int u^2 \exp\left(-\frac{u^2}{2}\right) du = \rho$$

('normal variance'), as required. //

This completes the identification of all five parameters in the bivariate normal distribution: two means μ_i, two variances σ_i^2, one correlation ρ.

We note in passing

Fact 8. The bivariate normal law has *elliptical contours.*

For, the contours are $Q(x,y) = const$, which are ellipses (as Galton found).

Moment Generating Function (MGF). Recall (see e.g. Haigh (2002), 102-6) $M(t)$, or $M_X(t)$, $:= E(e^{tX})$. For X normal $N(\mu, \sigma^2)$,

$$M(t) = \frac{1}{\sigma\sqrt{2\pi}} \int e^{tx} \exp(-\frac{1}{2}(x-\mu)^2/\sigma^2) dx.$$

Change variable to $u := (x-\mu)/\sigma$:

$$M(t) = \frac{1}{\sqrt{2\pi}} \int \exp(\mu t + \sigma u t - \frac{1}{2}u^2) du.$$

Completing the square,

$$M(t) = e^{\mu t} . \frac{1}{\sqrt{2\pi}} \int \exp(-\frac{1}{2}(u-\sigma t)^2) du . e^{\frac{1}{2}\sigma^2 t^2},$$

or $M_X(t) = \exp(\mu t + \frac{1}{2}\sigma^2 t^2)$ (recognising that the central term on the right is 1 – 'normal density') . So $M_{X-\mu}(t) = \exp(\frac{1}{2}\sigma^2 t^2)$. Then (check) $\mu = EX = M'_X(0)$, $var X = E[(X-\mu)^2] = M''_{X-\mu}(0)$.

Similarly in the bivariate case: the MGF is $M_{X,Y}(t_1, t_2) := E\exp(t_1 X + t_2 Y)$. In the bivariate normal case:

$$\begin{aligned} M(t_1, t_2) &= E(\exp(t_1 X + t_2 Y)) = \int\int \exp(t_1 x + t_2 y) f(x,y)\,dx\,dy \\ &= \int \exp(t_1 x) f_1(x)\,dx \int \exp(t_2 y) f(y|x)\,dy. \end{aligned}$$

The inner integral is the MGF of $Y|X = x$, which is $N(c_x, \sigma_2^2, (1-\rho^2))$, so is $\exp(c_x t_2 + \frac{1}{2}\sigma_2^2(1-\rho^2)t_2^2)$. By Fact 5 $c_x t_2 = [\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)]t_2$, so

$$M(t_1, t_2) = \exp(t_2\mu_2 - t_2\frac{\sigma_2}{\sigma_1}\mu_1 + \frac{1}{2}\sigma_2^2(1-\rho^2)t_2^2) \int \exp([t_1 + t_2\rho\frac{\sigma_2}{\sigma_1}]x) f_1(x)\,dx.$$

Since $f_1(x)$ is $N(\mu_1, \sigma_1^2)$, the inner integral is a normal MGF, which is thus $\exp(\mu_1[t_1 + t_2\rho\frac{\sigma_2}{\sigma_1}] + \frac{1}{2}\sigma_1^2[\ldots]^2)$. Combining the two terms and simplifying:
Fact 9. The joint MGF is

$$M_{X,Y}(t_1, t_2) = M(t_1, t_2) = \exp(\mu_1 t_1 + \mu_2 t_2 + \frac{1}{2}[\sigma_1^2 t_1^2 + 2\rho\sigma_1\sigma_2 t_1 t_2 + \sigma_2^2 t_2^2]).$$

Fact 10. X, Y are independent if and only if $\rho = 0$.

Proof. For densities: X, Y are independent iff the joint density $f_{X,Y}(x,y)$ *factorises* as the *product* of the marginal densities $f_X(x).f_Y(y)$ (see e.g. Haigh (2002), Cor. 4.17).

For MGFs: X, Y are independent iff the joint MGF $M_{X,Y}(t_1, t_2)$ *factorises* as the *product* of the marginal MGFs $M_X(t_1).M_Y(t_2)$. From Fact 9, this occurs iff $\rho = 0$. Similarly with CFs, if we prefer to work with them. //

Note. We can re-write Fact 5 above as

$$E[Y|X] = \mu_2 + \frac{\rho\sigma_1}{\sigma_2}(X - \mu_1).$$

So as $E[X] = \mu_1$, this illustrates the Conditional Mean Formula (II.4 Property 6, L10):

$$E[E[Y|X]] = \mu_2 + \frac{\rho\sigma_1}{\sigma_2}(E[X] - \mu_1) = \mu_2 = E[Y].$$

NHB