m3a22l30.tex

Lecture 30 18.12.2015

5. Real options (Investment options).

For background and details, see e.g.

[DP] Avinash K. DIXIT and Robert S. PINDYCK: *Investment under uncertainty*. Princeton University press, 1994;

[PS] G. PESKIR and A. N. SHIRYAEV: *Optimal stopping and free-boundary problems*. Birkhäuser, 2006.

The options considered above concern financial *derivatives* (so called because they derive from the underlying fundamentals such as stock). We turn now to options of another kind, concerned with business decision-making. Typically, we shall be concerned with the decision of whether or not to make a particular investment, and if so, when. Because these options concern the real economy (of manufacturing, etc.) rather than financial markets such as the stock market, such options are often called *real options*. But because they typically concern investment decisions, they are also often called *investment options*. There is a good introductory treatment in [DP].

The key features are as follows. We are contemplating making some major investment – buying or building a factory, drilling an oil well, etc. While if the decision goes wrong it may be possible to recoup some of the cost, much or most of it will usually be irrecoverable (a *sunk cost* – as with an oil well). So the investment is *irreversible* – at least in part. Just as stock prices are uncertain – so we model them as random, using some stochastic process – here too, the future profitability of the proposed investment is *uncertain*. Finally, we do not have to act now, or indeed at all. So we have an open-ended – or infinite – time-horizon, $T = \infty$.

We may choose to delay investment,

(a) to gather more information, to help us assess the project, or

(b) to continue to generate interest on the capital we propose to invest.

So we must recognize, and feed into the decision process, the value of *waiting for further information*. When we commit ourselves and make the decision to invest, it is not just the sunk cost that we lose – we lose the valuable option to wait for new information.

This situation is really that of an *American call* option with an infinite time-horizon. With such an American *call*, we have the right to buy at a specified price at a time of our choosing (or indeed, not to buy). Following Dixit & Pindyck [DP, Ch. 5], we formulate an *optimal stopping problem*, and solve it as a *free boundary problem*, using the *principle of smooth fit*.

We suppose the cost of the investment is I, and that the value of the project is given by a GBM, $X = (X_t) \sim GBM(\mu, \sigma)$ (the value of a project is uncertain for the same reasons that stock prices are uncertain; we model them both as stochastic processes; GBM is the default option here, just as in the BS theory of Ch. IV). If we invest at time τ, we want to maximize

$$V(X) := \max_{\tau} \ E[(X_\tau - I)e^{-r\tau}],$$

with r the riskless rate (discount rate) as before. Now if $\mu \leq 0$ the value of the project will fall, so we should invest immediately if $X_0 > I$ and not invest if not. If $\mu > r$, the growth of X will swamp the investment cost I and more than offset the discounting, so we should invest and there is no point in waiting. So we take $\mu \in (0, r]$. We invest iff the value x^* at the time of investment is *large enough*; finding x^* is part of the problem; x^* is a *free boundary* (between the continuation region and the investment region).

We need the following four conditions:

$$\frac{1}{2}\sigma^2 x^2 V''(x) + \mu x V'(x) - rV = 0, \qquad (i)$$

$$V(0) = 0, \qquad (ii)$$

$$V(x^*) = x^* - I, \qquad (iii)$$

$$V'(x^*) = 1 \qquad \text{(smooth pasting)}. \qquad (iv)$$

For (i): this comes from the *generator* of the diffusion $GBM(r, \sigma)$ (cf. the *SDE* for $GBM(r, \sigma)$, and Black-Scholes *PDE*, VI.2); for details, see [DP Ch. 5], or Peskir & Shiryaev [PS, Ch. III]. For (ii) ("Nothing will come of nothing"): the GBM does not hit 0, but if it approaches 0, so will the value of the project, so (ii) follows from this by continuity). For (iii), this is the *value-matching condition*: on investment, the firm receives the net pay-off $x^* - I$. For (iv) (*smooth pasting*): think of a rope stretched tightly over a convex surface.

The ODE (i) is *homogeneous* (cf. *Euler's theorem*). So we use a trial solution $V(x) = Cx^p$. So (i) gives that p satisfies the *fundamental quadratic*

$$Q(p) := \frac{1}{2}\sigma^2 p(p-1) + \mu p - r = 0.$$

The product of the roots is negative, and $Q(0) = -r < 0$, $Q(1) = \mu - r < 0$. So one root $p_1 > 1$ and the other $p_2 < 0$. The general solution is $V(x) =$

$C_1 x^{p_1} + C_2 x^{p_2}$, but from $V(0) = 0$, $C_2 = 0$, so $V(x) = C_1 x^{p_1}, = Cx^{p_1}$ say. With x^* the critical value at which it is optimal to invest, (iii) and (iv) give

$$V(x^*) = x^* - I, \qquad V'(x^*) = 1.$$

From these two equations, we can find C and x^*. The second is

$$V'(x^*) = Cp_1(x^*)^{p_1-1} = 1, \qquad C = (x^*)^{1-p_1}/p_1.$$

Then the first gives

$$C(x^*)^{p_1} = x^* - I, \qquad x^*/p_1 = x^* - I, \qquad x^* = \frac{p_1}{(p_1-1)} I.$$

The main feature here is the factor

$$q := p_1/(p_1 - 1) > 1$$

by which the value must exceed the investment cost I before investment should be made (q is used because this is related to "Tobin's q" in Economics). One can check that q increases with σ (the riskier the project, the more reluctant we are to invest), and also q increases with r (as then investing our capital risklessly becomes more attractive). Then the critical threshold above which it is optimal to invest is

$$x^* = qI.$$

Also

$$C = (qI)^{1-p_1}/p_1, \qquad V(x) = (qI)^{1-p_1} x^{p_1}/p_1.$$

The results above show that the traditional *net present value* (NPV – accountancy-based) approach to valuing real options is misleading – see [DP]. This is no surprise: our methods (*arbitrage pricing technique*, etc.) are superior to NPV!

Infinite time-horizon; American puts

The mathematics is similar [PS, Ch. III]. It turns out that the 'obvious guess' – sell the asset when its value becomes *too low* ($\leq b$, say, where b is the free boundary) is correct. One finds that the value process V has the form (with K the strike price and σ the volatility)

$$\begin{aligned} V(x) &= \frac{d}{r}\Big(\frac{K}{1+d/r}\Big)^{1+r/d} x^{-r/d} \qquad \text{if } x \in [b, \infty), \\ &= K - x \qquad \text{if } x \in (0, b], \quad \text{where} \quad b = K/(1 + \frac{1}{2}\sigma^2/r). \end{aligned}$$

Postscript.

1. One recent book on Financial Mathematics describes the subject as being composed of three strands:
arbitrage – the core economic concept, which we have used throughout;
martingales – the key probabilistic concept (Ch. III on);
numerics. Finance houses in the City use *models*, which they need to *calibrate to data* – a task involving both statistical and numerical skills, and in particular an ability to *programme*.

2. You will probably already have experience with at least one general mathematics package (e.g., Mathematica and/or Maple) (if not: get it, a.s.a.p.!). You may also know some Numerical Analysis, the theory behind computation. You may have encountered *simulation*, also known as *Monte Carlo*, and/or a branch of Probability and Statistics called *Markov Chain Monte Carlo (MCMC)* – computer-intensive methods for numerical solutions to problems too complicated to solve analytically. The leaders of R & D teams in the City need to be expert at both stochastic modelling (e.g., to propose new products), and simulation (to evaluate how these perform). Most of the ones I know use Matlab for this. At a lower level, quantitative analysts (quants) working under them need expertise in a computer language; C++ is the industry standard. If you are thinking of a career in Mathematical Finance, learn C++, as soon as possible, and for academic credit.

3. This course deals with *equity markets* – with *stocks*, and financial derivatives of them – options on stocks, etc. The relevant mathematics is *finite-dimensional*. Lurking in the background are *bond markets* ('money markets': bonds, gilts etc., where *interest rates* dominate), and the relevant options – *interest-rate derivatives*, and *foreign exchange* between different currencies ('forex'). The resulting mathematics (which is highly topical, and so in great demand in the City!) is *infinite-dimensional*, and so much harder than the equity-market theory we have done. However, the underlying principles are basically the same. One has to learn to walk before one learns to run, and equity markets serve as a preparation for money markets.

4. The aim of this lecture course is simple. It is to familiarize the student with the basics of Black-Scholes theory, as the core of modern finance, and with the mathematics necessary to understand this. The motivation driving the ever-increasing study of this material is the financial services industry and the City. I hope that any of you who seek City careers will find this introduction to the subject useful in later life. NHB, 2015