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Lecture 3. 16.10.2015

Black, Scholes and Merton

As everywhere, triumph and disaster can always happen, and one has to use common sense. Triumph: Scholes and Merton were awarded the Nobel Prize for Economics in 1997 (Black died in 1995, and the prize cannot be awarded posthumously). Disaster: Scholes and Merton were on the board of the hedge fund Long Term Capital Management, which ignominiously collapsed with enormous losses in 1998. Pushing a good theory too far – beyond all sensible limits – is asking for trouble, even if one invented the theory and got the Nobel Prize for it, and if one asks for trouble, one can expect to get it.

4. Markets and Options.

Markets.

This course is about the mathematics of *financial markets*. Types:
Stock markets [New York, London, ...], dealing in stocks/shares/equities, etc.,
Bond markets, dealing in government bonds (gilts, ...),
Currency or foreign exchange ('forex') markets,
Futures and *options markets*, dealing in financial instruments derived from the above - *financial derivatives* such as *options* of various types.

Options.

Economic activity, and trading, involves *risk*. One may have to, or choose to, make a judgement involving committing funds ('taking a position') based on prediction of the future in the presence of uncertainty. With hindsight, one might or might not regret taking that position. An *option* is a financial instrument giving one the *right but not the obligation* to make a specified transaction at (or by) a specified date at a specified price. Whether or not the option will be exercised depends on (is contingent on) the uncertain future, so is also known as a *contingent claim*.

Types of option.

Call options give one the right (but not – without further comment now – the obligation) to *buy*. [Remember: to *buy* something, one *calls* for it.]
Put options give one the right to *sell*. [Remember: to *sell* something, one *puts* it on the market.]
European options give one the right to buy/sell *on* the specified date, the *expiry* date, when the option *expires* or matures.
American options give one the right to buy/sell *at any time* prior to or at expiry. Thus:

European options: exercise *at* expiry,

American options: exercise *by* expiry.

Note. The terms European, American (Asian, etc.) refer only to the type of option, and no longer bear any relation to the area in the name. Most options traded worldwide these days are American.

History. As discussed in §1, over-the-counter (OTC) options were long ago negotiated by a broker between a buyer and a seller. Then in 1973 (the year of the Black-Scholes formula, the central result of the course), the Chicago Board Options Exchange (CBOE) began trading in options on some stocks. Since then, the growth of options has been explosive. Options are now traded on all the major world exchanges, in enormous volumes. Often, the market in derivatives is *much larger* than the market in the underlying assets – an important source of instability in financial markets.

The simplest call and put options are now so standard they are called *vanilla* options. Many kinds of options now exist, including so-called *exotic* options (Asian, barrier, etc.) – on which you need a separate course. For *real options* (also called *investment options*) – see L30.

Terminology.

The asset to which the option refers is called the *underlying asset* or the *underlying*. The price at which the transaction to buy/sell the underlying, on/by the expiry date (if exercised), is called the *exercise price* or *strike price*. We shall usually use K for the strike price, time $t = 0$ for the initial time (when the contract between the buyer and the seller of the option is struck), time $t = T$ for the expiry or final time.

Consider a European *call* option, with strike price K and underlying worth S_t at time t. If $S_T > K$, the option is *in the money*: the holder will/should *exercise* the option, obtaining an asset worth S_T $(> K)$ for K. He can immediately sell the asset for S_T, making a *profit* of $S_T - K$ (> 0).

If $S_T = K$, the option is said to be *at the money*.

If $S_T < K$, the option is *out of the money*, and should not be exercised. It is worthless, and is thrown away.

The *pay-off* from the option is thus

$$S_T - K \text{ if } S_T > K, \quad 0 \quad \text{otherwise},$$

which may be written more briefly as

$$max(S_T - K, 0) \text{ or } (S_T - K)_+$$

$(x_+ := max(x, 0), x_- := -min(x, 0); x = x_+ - x_-, |x| = x_+ + x_-; (-x)_+ = max(-x, 0) = -min(x, 0) = x_-)$.
Similarly, the payoff from a *put* option is

$$K - S_T \text{ if } S_T \leq K, \qquad 0 \text{ if } S_T > K,$$

or $(K - S_T)_+$.
Option pricing.

The fundamental problem in the mathematics of options is that of *option pricing*. The modern theory began with the *Black-Scholes formula* for pricing European options in 1973. We shall deal with the Black-Scholes theory, and cover the pricing of European options in full. We also discuss American options: these are harder, and lack explicit formulae such as the Black-Scholes formula; consequently, one needs to evaluate them numerically. The pricing of Asian options is even harder.
Perfect Markets. For simplicity, we shall confine ourselves to option pricing in the simplest (idealised) case, of a *perfect*, or *frictionless*, market. First, there are no *transaction costs* (one can include transaction costs in the theory, but this is considerably harder). Similarly, we assume that interest rates for borrowing and for lending are the same (which is unrealistic, as banks make their money on the difference), and also that all traders have access to the same – perfect – information about the past history of price movements, but have no foreknowledge of price-sensitive information (i.e. no insider trading). We shall assume no restrictions on *liquidity* – that is, one can buy or sell unlimited quantities of stock at the currently quoted price. That is, our economic agents are *price takers* and not *price makers*. (This comes back to §1 on the relationship between Economics and Finance. In practice, big trades do move markets. Also, in a crisis, no-one wants to trade, and liquidity dries up – basically, this is what did for LTCM.) In practice, very small trades are not economic (the stockbroker may only deal in units of reasonable size, etc.). We shall ignore all these complications for the sake of simplicity.

5. Portfolios and Hedging.

Portfolios.

We consider an investor with capital to invest. The simplest model is that in which he has two (or more) choices: to invest in
(i) a *bank account* – assumed *riskless*, and yielding interest. For simplicity, we assume the interest rate is a constant $r > 0$ (usually called the *short rate* of interest: interest rates may be different outside $[0, T]$); thus \$$B$ invested

at time t grows to $\$Be^{r(T-t)}$ by time t;
(ii) one (or more) *risky assets* or stocks, whose value at time t is S_t (scalar or vector).
A *portfolio* is a division (B_t, S_t) of the investor's capital between bank account and stock holdings at time t.
A *trading strategy* is a rule (suitably restricted – see Chapters IV and VI) chosen by the investor to update his portfolio over time as new price information on the risky stock(s) comes in.
Hedging. Hedging is an attempt to reduce exposure to risk by adopting opposite positions – e.g., in holding both call and put options in the same underlying, or by adjusting a portfolio as above to cover possible losses.
Why buy options? Options have two main uses: speculation and hedging. In speculation, available funds ('hot money') are invested opportunistically in the hope of making a profit: the underlying itself is irrelevant to the investor (speculator), who is only interested in the potential for possible profit that trade involving it may present. Hedging, by contrast, is typically engaged in by companies who have to deal habitually in intrinsically risky assets such as foreign exchange next year, wheat/copper/oil next year, etc. This protects their economic base (trade in wheat/copper/oil, or manufacture of products using these as raw materials), and lets them focus their effort in their chosen area of trade or manufacture. But for speculators, it is the market (forex, commodities or whatever) itself which is their main focus.

Because the value of an option at expiry is so sensitive to price – it reflects movements in the price of the underlying in exaggerated form – the holding (or trading) of options and other derivatives presents greater opportunities for profit (and indeed, for loss) than trade in the underlying (this is why speculators buy options!). They are correspondingly more risky.

One of the main insights of the fundamental work of Black and Scholes was that one can (at least in the most basic model) *hedge* against meeting a contingent claim by *replicating* it: constructing a portfolio, adjusted or rebalanced as time unfolds and new price information comes in, whose pay-off *is* the amount of the contingent claim.

6. Arbitrage.

Economic agents go to the market for various reasons. One the one hand, companies may wish to insure, or *hedge*, against adverse price movements that might affect their core business. On the other hand, *speculators* may be uninterested in the specific economic background, but only interested in making a profit from some financial transaction. The relation between hedging ('good') and speculation ('bad') is to some extent symbiotic (one cannot lay off a risk unless someone else is prepared to take it on, and why should he unless he expects to make money by doing so). Nevertheless, one feels that it should not be possible to extract money from the market without genuinely engaging in it, by taking *risk*: all business activity is risky. Indeed, were it possible to do so, people would do so – in unlimited quantities, thus sucking money parasitically out of the market, using it as a 'money-pump'. This would undermine the stability and viability of the market in the long run – and in particular, make it impossible for the market to be in *equilibrium*.

The view we take of modelling markets as NA is not that arbitrage opportunities do not exist, but that if they do exist in any quantity, people will rush to exploit them, and thereby dissipate them – 'arbitrage them away'.

Financial markets involve both riskless (bank account) and risky (stocks, etc.) assets. For investors, the only point of taking risk is the chance of a greater profit than the riskless procedure of putting one's money in the bank (the mathematics of which – compound interest – does not require a degree or MSc course!). In general, the greater the risk, the greater the return required to make investment an attractive enough prospect to attract funds.

It is usually better to work, not in face-value or nominal terms, but in *discounted* terms, allowing for the exponential growth-rate e^{rt} of risklessly invested money. So, profit and loss are generally reckoned against this discounted benchmark. So a market with arbitrage opportunities would be a disorderly market – too disorderly to model. The remarkable thing is the converse. It turns out that the minimal requirement of absence of arbitrage opportunities is enough to allow one to build a model of a financial market which – while admittedly idealised (frictionless market – no transaction costs, etc.) – is realistic enough both to provide real insight and to handle the mathematics necessary to price standard options (Black-Scholes theory). We shall see that arbitrage arguments suffice to determine prices – the *arbitrage pricing technique* (APT – S. A. Ross, 1976, 1978).

Short-selling.

Just as we can borrow money from the bank, in many markets, risky assets such as stocks may be treated in the same way. We may have a positive or zero holding – or a *negative holding* (notionally borrowing stock, which we will be obliged to repay – or repay its current value). In particular, we may be allowed to *sell stock we do not own*. This is called *short-selling*, and is perfectly legal (subject to appropriate regulation) in many markets. Think of short-selling as *borrowing*.