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Lecture 29 15.12.2015

§4. Girsanov's Theorem

Consider first ([KS], §3.5) independent $N(0,1)$ random variables $Z_1, \cdots, Z_n$ on $(\Omega, \mathcal{F}, \mathcal{P})$. Given a vector $\mu = (\mu_1, \cdots, \mu_n)$, consider a new probability measure $\tilde{P}$ on $(\Omega, \mathcal{F})$ defined by

$$\tilde{P}(d\omega) = \exp\{\Sigma_1^n \mu_i Z_i(\omega) - \frac{1}{2}\Sigma_1^n \mu_i^2\}.P(d\omega).$$

This is a positive measure as $\exp\{.\} > 0$, and integrates to 1 as $\int \exp\{\mu_i Z_i\} dP = \exp\{\frac{1}{2}\mu_i^2\}$ (normal MGF), so is a probability measure. It is also equivalent to P (has the same null sets – actually, the only null set are Lebesgue-null sets, in each case), again as the exponential term is positive. Also

$$\tilde{P}(Z_i \in dz_i, \quad i = 1, \cdots, n) = \exp\{\Sigma_1^n \mu_i z_i - \frac{1}{2}\Sigma_1^n \mu_i^2\}.P(Z_i \in dz_i, \quad i = 1, \cdots, n)$$

($Z_i \in dz_i$ means $z_i \le Z_i \le z_i + dz_i$, so here $Z_i = z_i$ to first order)

$$= (2\pi)^{-\frac{1}{2}n} \exp\{\Sigma \mu_i z_i - \frac{1}{2}\Sigma \mu_i^2 - \frac{1}{2}\Sigma z_i^2\}\Pi dz_i$$
$$= (2\pi)^{-\frac{1}{2}n} \exp\{-\frac{1}{2}\Sigma (z_i - \mu_i)^2\} dz_1 \cdots dz_n.$$

This says that if the Z_i are independent $N(0,1)$ under P, they are independent $N(\mu_i, 1)$ under $\tilde{P}$. Thus the effect of the *change of measure* $P \mapsto \tilde{P}$, from the original measure P to the *equivalent* measure $\tilde{P}$, is to *change the mean*, from $0 = (0, \cdots, 0)$ to $\mu = (\mu_1, \cdots, \mu_n)$.

This result extends to infinitely many dimensions – i.e., stochastic processes. We quote (Igor Vladimirovich GIRSANOV (1934-67) in 1960):

Theorem (Girsanov's Theorem). Let $(\mu_t : 0 \le t \le T)$ be an adapted process with $\int_0^T \mu_t^2 dt < \infty \quad a.s.$ such that the process L with

$$L_t := \exp\{\int_0^t \mu_s dW_s - \frac{1}{2}\int_0^t \mu_s^2 ds\} \qquad (0 \le t \le T)$$

is a martingale. Then, under the probability P_L with density L_T relative to P, the process W^* defined by

$$W_t^* := W_t - \int_0^t \mu_s ds, \qquad (0 \le t \le T)$$

is a standard Brownian motion (so W is BM + $\int_0^t \mu_s ds$).

Here, L_t is the *Radon-Nikodym derivative* of P_L w.r.t. P on the σ-algebra $\mathcal{F}_t$. In particular, for $\mu_t \equiv \mu$, change of measure by introducing the RN derivative $\exp\{\mu W_t - \frac{1}{2}\mu^2\}$ corresponds to a change of drift from 0 to μ.

The martingale condition in Girsanov's theorem is satisfied in the case μ_t constant, from the SDE for GBM (VI.1, L25).

Girsanov's Theorem (or the Cameron-Martin-Girsanov Theorem) is formulated in varying degrees of generality, and proved, in [KS, §3.5], [RY, VIII].

The Sharpe ratio.

There is no point in investing in a risky asset with mean return rate μ, when cash is a riskless asset with return rate r, unless $\mu > r$. The excess return $\mu - r$ (the investor's reward for taking a risk) is compared with the risk, as measured by the volatility σ, via the *Sharpe ratio*

$$\theta := (\mu - r)/\sigma,$$

also known as the *market price of risk*. This is important, both here (see below), in CAPM (I.3, L2), and in asset allocation decisions.

Consider now the Black-Scholes model, with dynamics

$$dB_t = rB_t dt, \qquad dS_t = \mu S_t dt + \sigma S_t dW_t.$$

The discounted asset prices $\tilde{S}_t := e^{-rt} S_t$ have dynamics given, as before, by

$$\begin{aligned} d\tilde{S}_t &= -re^{-rt} S_t dt + e^{-rt} dS_t \\ &= -r\tilde{S}_t dt + \mu \tilde{S}_t dt + \sigma \tilde{S}_t dW_t \\ &= (\mu - r)\tilde{S}_t dt + \sigma \tilde{S}_t dW_t \\ &= \sigma \tilde{S}_t (\theta dt + dW_t). \end{aligned}$$

Now the drift – θdt – term here prevents $\tilde{S}_t$ being a martingale; the noise – dW_t – term gives a stochastic integral, a martingale. Girsanov's theorem suggests the change of measure $P \mapsto P^*$ to the EMM (or risk-neutral measure) – $\mu \mapsto r$, $\theta \mapsto 0$ – making the discounted asset price a martingale. This
(i) gives directly the continuous-time version of the FTAP/RNVF: *to price assets, take expectations of discounted prices under the risk-neutral measure* (see below for completeness and uniqueness of EMM and prices);
(ii) allows a probabilistic treatment of the Black-Scholes model, avoiding the

detour via PDEs of §2, §3.

Theorem (Representation Theorem for Brownian Martingales). Let $(M_t : 0 \leq t \leq T)$ be a square-integrable martingale with respect to the Brownian filtration $(\mathcal{F}_t)$. Then there exists an adapted process $H = (H_t : 0 \leq t \leq T)$ with $E \int H_s^2 ds < \infty$ such that

$$M_t = M_0 + \int_0^t H_s dW_s, \qquad 0 \leq t \leq T.$$

That is, all Brownian martingales may be represented as stochastic integrals with respect to Brownian motion.

We refer to, e.g., [KS], [RY] for proof.

The economic relevance of the Representation Theorem is that it shows (see e.g. [KS, I.6], and below) that the Black-Scholes model is *complete* – that is, that EMMs are unique, and so that *Black-Scholes prices are unique* (we know this already, from FTAP/RNVF, VI.3 L28). Mathematically, the result is purely a consequence of properties of the Brownian filtration. The desirable mathematical properties of BM are thus seen to have hidden within them desirable economic and financial consequences of real practical value.

To summarise the basic case (μ and σ constant) in a nutshell:
(i) Dynamics are given by GBM, $dS_t = \mu S dt + \sigma S dW_t$.
(ii) Discount: $d\tilde{S}_t = (\mu - r)\tilde{S} dt + \sigma \tilde{S} dW_t = \sigma \tilde{S}(\theta dt + dW_t)$.
(iii) Use Girsanov's Theorem to change μ to r, so $\theta := (\mu - r)/\sigma$ to 0: under P^*, $d\tilde{S}_t = \sigma \tilde{S} dW_t$.
(iv) Integrate: the RHS gives a P^*-martingale, so has constant E^*-expectation.
(v) Hence the RNVF (VI.3, L28).
(vi) Hence the BS formula, by integration (L28), for European calls and puts.
Hedging.

To find a hedging strategy $H = (H_t^0, H_t)$ (H_t^0 for cash, H_t for stock) that replicates the value process $V = (V_t)$, itself given by RNVF (VI.3 L28):

$$V_t = H_t^0 + H_t S_t = E^*[e^{-r(T-t)} h | \mathcal{F}_t].$$

Now

$$M_t := E^*[e^{-rT} h | \mathcal{F}_t]$$

is a martingale (indeed, a uniformly integrable mg: IV.4 L12, V.2 L21) under the filtration $\mathcal{F}_t$, that of the driving BM in (GBM) (VI.1 L25, VI.2 L26), and the filtration is unchanged by the Girsanov change of measure (we quote this). So by the Representation Theorem for Brownian Martingales, there is some adapted process $K = (K_t)$ with

$$M_t = M_0 + \int_0^t K_s dW_s \qquad (t \in [0, T]).$$

Take

$$H_t := K_t/(\sigma \tilde{S}_t), \qquad H_t^0 := M_t - H_t \tilde{S}_t.$$

Then

$$dM_t = K_t dW_t = \frac{K_t}{\sigma \tilde{S}_t}.\sigma \tilde{S}_t dW_t = H_t d\tilde{S}_t,$$

and the strategy given by K is self-financing, by VI.2 L26. This is of limited practical value:
(a) the Representation Th. does not give $K = (K_t)$ explicitly – it is merely an existence proof;
(b) we already know that, as Brownian paths have infinite variation, exact hedging in the Black-Scholes model is too rough to be practically possible.
Comments.
1. Calculation. When solutions have to be found numerically (as is the case in general – though not for some important special cases such as European calls and puts, as in BS), we again have a choice of
(i) analytic methods: numerical solution of a PDE,
(ii) probabilistic methods: evaluation, by RNVF, of an expectation.
A comparison of convenience between these two methods depends on one's experience of numerical computation and the software available.
2. Discrete and continuous time. One often has a choice between discrete and continuous time. For discrete time, we have proved everything; for continuous time, we have had to quote the hard proofs. Note that in continuous time we can use calculus – PDEs, SDEs, Itô calculus, etc. In discrete time we use instead the calculus of finite differences.
3. American options. The situation here is much as in discrete time. It is important: most options traded are American. There are no explicit formulae; in continuous time one has to discretise to reduce to the discrete-time case, and solve by backward recursion as in Ch. IV. The value of the American option splits, into two components: the *intrinsic value* (European value), and the *early-exercise premium.* The relevant mathematics here involves the *Riesz decomposition* of Potential Theory.