m3a22l28tex

Lecture 28 14.12.2015

Taking existence of a unique solution for granted for the moment, consider a smooth function $F(s, X_s)$ of it. By Itô's Lemma,

$$dF = F_1 ds + F_2 dX + \frac{1}{2} F_{22} (dX)^2,$$

and as $(dX)^2 = (\mu ds + \sigma dW_s)^2 = \sigma^2 (dW_s)^2 = \sigma^2 ds$, this is

$$dF = F_1 ds + F_2(\mu ds + \sigma dW_s) + \frac{1}{2}\sigma^2 F_{22} ds = (F_1 + \mu F_2 + \frac{1}{2}\sigma^2 F_{22}) ds + \sigma F_2 dW_s. \qquad (*)$$

Now suppose that F satisfies the PDE, with boundary condition (BC),

$$F_1(t,x) + \mu(t,x) F_2(t,x) + \frac{1}{2}\sigma^2 F_{22}(t,x) = g(t,x) \qquad (PDE)$$

$$F(T,x) = h(x). \qquad (BC)$$

Then $(*)$ gives

$$dF = g ds + \sigma F_2 dW_s,$$

which can be written in stochastic-integral form as

$$F(T, X_T) = F(t, X_t) + \int_t^T g(s, X_s) ds + \int_t^T \sigma(s, X_s) F_2(s, X_s) dW_s.$$

The stochastic integral on the right is a martingale, so has constant expectation, which must be 0 as it starts at 0. Recalling that $X_t = x$, writing $E_{t,x}$ for expectation with value x and starting-time t, and the price at expiry T as $h(X_T)$ as before, taking $E_{t,x}$ gives

$$E_{t,x} h(X_T) = F(t,x) + E_{t,x} \int_t^T g(s, X_s) ds.$$

This gives:

Theorem (Feynman-Kac Formula). The solution $F = F(t,x)$ to the PDE

$$F_1(t,x) + \mu(t,x) F_2(t,x) + \frac{1}{2}\sigma^2(t,x) F_{22}(t,x) = g(t,x) \qquad (PDE)$$

with final condition $F(T, x) = h(x)$ has the stochastic representation

$$F(t, x) = E_{t,x} h(X_T) - E_{t,x} \int_t^T g(s, X_s) ds, \qquad (FK)$$

where X satisfies the SDE

$$dX_s = \mu(s, X_s) ds + \sigma(s, X_s) dW_s \qquad (t \le s \le T) \qquad (SDE)$$

with initial condition $X_t = x$.

Now replace $\mu(t, x)$ by rx, $\sigma(t, x)$ by σx, g by rF in the Feynman-Kac formula above. The SDE becomes

$$dX_s = rX_s ds + \sigma X_s dW_s \qquad (**)$$

– *the same as for a risky asset with mean return-rate* r (the short interest-rate for a riskless asset) *in place of* μ (which disappeared in the Black-Scholes result). The PDE becomes

$$F_1 + rxF_2 + \frac{1}{2}\sigma^2 x^2 F_{22} = rF, \qquad (BS)$$

the Black-Scholes PDE. So by the Feynman-Kac formula,

$$dF = rF ds + \sigma F_2 dW_s, \qquad F(T, s) = h(s).$$

We can eliminate the first term on the right by discounting at rate r: write $G(s, X_s) := e^{-rs} F(s, X_s)$ for the discounted price process. Then as before,

$$dG = -re^{-rs} F ds + e^{-rs} dF = e^{-rs}(dF - rF ds) = e^{-rs}.\sigma F_2 dW.$$

Then integrating, G is a stochastic integral, so a martingale: *the discounted price process* $G(s, X_s) = e^{-rs} F(s, X_s)$ *is a martingale*, under the measure P^* giving the dynamics in $(**)$. This is the measure P we started with, *except* that μ has been changed to r. Thus, G has constant P^*-expectation:

$$E^*_{t,x} G(t, X_t) = E^*_{t,x} e^{-rt} F(t, X_t) = e^{-rt} F(t, x) = E^*_{T,x} e^{-rT} F(T, X_T) = e^{-rT} h(X_T).$$

This gives the following result:

Theorem (Risk-Neutral Valuation Formula). The no-arbitrage price of the claim $h(S_T)$ with payoff function h is given by

$$F(t,x) = e^{-r(T-t)} E^*_{t,x} h(S_T),$$

where $S_t = x$ is the asset price at time t and P^* is the measure under which the asset price dynamics are given by

$$dS_t = rS_t dt + \sigma(t, S_t) S_t dW_t.$$

Corollary. In the Black-Scholes model above, the arbitrage-free price does not depend on the mean return rate μ of the underlying asset.

Comments.
1. *Risk-neutral measure.*
We call P^* the *risk-neutral* probability measure. It is equivalent to P (by Girsanov's Theorem – the change-of-measure result, which deals with change of drift in SDEs – see VI.4, L29 below), and is a martingale measure (as the discounted asset prices are P^*-martingales, by above), i.e. P^* (or Q) is the *equivalent martingale measure (EMM).*
2. *Fundamental Theorem of Asset Pricing (FTAP); Risk-Neutral Valuation Formula (RNVF).*
The above continuous-time result may be summarised just as the FTAP/RNVF in discrete time: to get the no-arbitrage price of a contingent claim, take the *discounted expected value under the equivalent mg (risk-neutral) measure.*
3. *Completeness.*
In discrete time, we saw that absence of arbitrage corresponded to *existence* of risk-neutral measures, completeness to *uniqueness*. We have obtained existence and uniqueness here (and so completeness), by appealing to existence and uniqueness theorems for PDEs (which we have not proved!). A more probabilistic route is to use Girsanov's Theorem (VI.4) instead. Completeness questions then become questions on representation theorems for Brownian martingales (VI4). As usual, there is a choice of routes to the major results – in this case, a trade-off between analysis (PDEs) and probability (Girsanov's Theorem and the Representation Theorem for Brownian Martingales, VI.4).

Now the process specified under P^* by the dynamics $(**)$ is our old friend geometric Brownian motion, $GBM(r, \sigma)$. Thus if S_t has P^*-dynamics

$$dS_t = rS_t dt + \sigma S_t dW_t, \qquad S_t = s,$$

with W a P^*-Brownian motion, then we can write S_T explicitly as

$$S_T = s \exp\{(r - \frac{1}{2}\sigma^2)(T - t) + \sigma(W_T - W_t)\}.$$

Now $W_T - W_t$ is normal $N(0, T-t)$, so $(W_T - W_t)/\sqrt{T-t} =: Z \sim N(0,1)$:

$$S_T = s \exp\{(r - \frac{1}{2}\sigma^2)(T - t) + \sigma Z \sqrt{T-t}\}, \qquad Z \sim N(0,1).$$

So by the Risk-Neutral Valuation Formula, the pricing formula is

$$F(t,x) = e^{-r(T-t)} \int_{-\infty}^{\infty} h(s \exp\{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma(T-t)^{\frac{1}{2}} x\}).\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} dx.$$

For a general payoff function h, there is no explicit formula for the integral, which has to be evaluated numerically. But we can evaluate the integral for the basic case of a European call option with strike-price K:

$$h(s) = (s - K)^+.$$

Then

$$F(t,x) = e^{-r(T-t)} \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} [s \exp\{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma(T-t)^{\frac{1}{2}} x\} - K]_+ dx.$$

We have already evaluated integrals of this type in Chapter IV, where we obtained the Black-Scholes formula from the binomial model by a passage to the limit. Completing the square in the exponential as before gives the

Continuous Black-Scholes Formula.

$$F(t,s) = s\Phi(d_+) - e^{-r(T-t)} K \Phi(d_-),$$

where

$$d_{\pm} := [\log(s/K) + (r \pm \frac{1}{2}\sigma^2)(T-t)]/\sigma\sqrt{T-t}.$$