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Lecture 24 4.12.2015

Note. The stochastic integral for simple integrands is essentially a martingale transform, and the above is essentially the proof of Ch. III that martingale transforms are martingales.

We pause to note a property of martingales which we shall need below. Call $X_t - X_s$ the *increment* of X over $(s,t]$. Then for a *martingale* X, *the product of the increments over disjoint intervals has zero mean.* For, if $s < t \leq u < v$,

$$\begin{aligned} E[(X_v - X_u)(X_t - X_s)] &= E[E[(X_v - X_u)(X_t - X_s)|\mathcal{F}_u]] \\ &= E[(X_t - X_s)E[(X_v - X_u)|\mathcal{F}_u]], \end{aligned}$$

taking out what is known (as $s, t \leq u$). The inner expectation is zero by the martingale property, so the LHS is zero, as required.

D (*Itô isometry*). $E[(I_t(X))^2]$, or $E[(\int_0^t X_s dB_s)^2]$, $= E[\int_0^t X_s^2 ds]$.
Proof. The LHS above is $E[I_t(X).I_t(X)]$, i.e.

$$E[(\Sigma_{i=0}^{n-1}\xi_i(B(t_{i+1}) - B(t_i)) + \xi_n(B(t) - B(t_n)))^2].$$

Expanding the square, the cross-terms have expectation zero by above, so

$$E[\Sigma_{i=0}^{n-1}\xi_i^2(B(t_{i+i} - B(t_i))^2 + \xi_n^2(B(t) - B(t_n))^2].$$

Since ξ_i is $\mathcal{F}_{t_i}$-measurable, each ξ_i^2-term is independent of the squared Brownian increment term following it, which has expectation $var(B(t_{i+1}) - B(t_i)) = t_{i+1} - t_i$. So we obtain

$$\Sigma_{i=0}^{n-1} E[\xi_i^2](t_{i+1} - t_i) + E[\xi_n^2](t - t_n).$$

This is $\int_0^t E[X_u^2]du = E[\int_0^t X_u^2 du]$, as required.

E. *Itô isometry (continued).* $I_t(X) - I_s(X) = \int_s^t X_u dB_u$ satisfies

$$E[(\int_s^t X_u dB_u)^2] = E[\int_s^t X_u^2 du] \qquad P - a.s.$$

Proof: as above.

F. *Quadratic variation.* The QV of $I_t(X) = \int_0^t X_u dB_u$ is $\int_0^t X_u^2 du$.

This is proved in the same way as the case $X \equiv 1$, that B has quadratic variation process t.

Integrands.

The properties above suggest that $\int_0^t X dB$ should be defined only for processes with

$$\int_0^t E[X_u^2] du < \infty \qquad \text{for all} \quad t.$$

We shall restrict attention to such X in what follows. This gives us an L_2-theory of stochastic integration (compare the L_2-spaces introduced in Ch. II), for which Hilbert-space methods are available.

3. *Approximation.*

Recall steps 1 (indicators) and 2 (simple integrands). By analogy with the integral of Ch. II, we seek a suitable class of integrands suitably approximable by simple integrands. It turns out that:

(i) The suitable class of integrands is the class of left-continuous adapted processes X with $\int_0^t E[X_u^2] du < \infty$ for all $t > 0$ (or all $t \in [0, T]$ with finite time-horizon T, as here),

(ii) Each such X may be approximated by a sequence of simple integrands X_n so that the stochastic integral $I_t(X) = \int_0^t X dB$ may be defined as the limit of $I_t(X_n) = \int_0^t X_n dB$,

(iii) The stochastic integral $\int_0^t X dB$ so defined still has properties A-F above.

It is not possible to include detailed proofs of these assertions in a course of this type [recall that we did not construct the measure-theoretic integral of Ch. II in detail either - and this is harder!]. The key technical ingredient needed is the *Kunita-Watanabe inequalities.* See e.g. [KS], §§3.1-2.

One can define stochastic integration in much greater generality.

1. *Integrands.* The natural class of integrands X to use here is the class of *predictable* processes. These include the left-continuous processes to which we confine ourselves above.

2. *Integrators.* One can construct a closely analogous theory for stochastic integrals with the Brownian integrator B above replaced by a *continuous local martingale* integrator M (or more generally by a *local martingale*: see

below). The properties above hold, with D replaced by

$$E[(\int_0^t X_u dM_u)^2] = E[\int_0^t X_u^2 d\langle M\rangle_u].$$

See e.g. [KS], [RY] for details.

One can generalise further to *semimartingale* integrators: these are processes expressible as the sum of a local martingale and a process of (locally) finite variation. Now C is replaced by: stochastic integrals of local martingales are local martingales. See e.g. [RW1] or Meyer (1976) for details.

§6. Stochastic Differential Equations (SDEs) and Itô's Lemma

Suppose that U, V are adapted processes,with U locally integrable (so $\int_0^t U_s ds$ is defined as an ordinary integral, as in Ch. II), and V is left-continuous with $\int_0^t E[V_u^2]du < \infty$ for all t (so $\int_0^t V_s dB_s$ is defined as a stochastic integral, as in §5). Then

$$X_t := x_0 + \int_0^t U_s ds + \int_0^t V_s dB_s$$

defines a stochastic process X with $X_0 = x_0$. It is customary, and convenient, to express such an equation symbolically in differential form, in terms of the *stochastic differential equation*

$$dX_t = U_t dt + V_t dB_t, \qquad X_0 = x_0. \qquad (SDE)$$

Now suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is a function, continuously differentiable once in its first argument (which will denote time), and twice in its second argument (space): $f \in C^{1,2}$. The question arises of giving a meaning to the stochastic differential $df(t, X_t)$ of the process $f(t, X_t)$, and finding it.

Recall the Taylor expansion of a smooth function of several variables, $f(x_0, x_1, \cdots, x_d)$ say. We use suffices to denote partial derivatives: $f_i := \partial f/\partial x_i$, $\quad f_{i,j} := \partial^2 f/\partial x_i \partial x_j$ (recall that if partials not only exist but are continuous, then the order of partial differentiation can be changed: $f_{i,j} = f_{j,i}$, etc.). Then for $x = (x_0, x_1, \cdots, x_d)$ near u,

$$f(x) = f(u) + \Sigma_{i=0}^d (x_i - u_i) f_i(u) + \frac{1}{2}\Sigma_{i,j=0}^d (x_i - u_i)(x_j - u_j) f_{i,j}(u) + \cdots$$

In our case (writing t_0 in place of 0 for the starting time):

$$f(t,X_t) = f(t_0,X(t_0)) + (t-t_0)f_1(t_0,X(t_0)) + (X(t)-X(t_0))f_2 + \frac{1}{2}(t-t_0)^2 f_{11} +$$

$$(t-t_0)(X(t)-X(t_0))f_{12} + \frac{1}{2}(X(t)-X(t_0))^2 f_{22} + \cdots,$$

which may be written symbolically as

$$df(t,X(t)) = f_1 dt + f_2 dX + \frac{1}{2} f_{11}(dt)^2 + f_{12} dt dX + \frac{1}{2} f_{22}(dX)^2 + \cdots.$$

In this, we
(i) substitute $dX_t = U_t dt + V_t dB_t$ from above,
(ii) substitute $(dB_t)^2 = dt$, i.e. $|dB_t| = \sqrt{dt}$, from §4:

$$df = f_1 dt + f_2(Udt+VdB) + \frac{1}{2} f_{11}(dt)^2 + f_{12} dt(Udt+VdB) + \frac{1}{2} f_{22}(Udt+VdB)^2 + \cdots$$

Now using $(dB)^2 = dt$,

$$\begin{aligned}(Udt+VdB)^2 &= V^2 dt + 2UV dt dB + U^2(dt)^2 \\ &= V^2 dt + \text{higher-order terms} :\end{aligned}$$

$$df = (f_1 + Uf_2 + \frac{1}{2}V^2 f_{22})dt + Vf_2 dB + \text{higher-order terms.}$$

Summarising, we obtain *Itô's Lemma*, the analogue for the Itô or stochastic calculus of the chain rule for ordinary (Newton-Leibniz) calculus:

Theorem (Itô's Lemma). If X_t has stochastic differential

$$dX_t = U_t dt + V_t dB_t, \qquad X_0 = x_0,$$

and $f \in C^{1,2}$, then $f = f(t,X_t)$ has stochastic differential

$$df = (f_1 + Uf_2 + \frac{1}{2}V^2 f_{22})dt + Vf_2 dB_t.$$

That is, writing f_0 for $f(0,x_0)$, the initial value of f,

$$f(t,X_t)) = f_0 + \int_0^t (f_1 + Uf_2 + \frac{1}{2}V^2 f_{22})dt + \int_0^t Vf_2 dB.$$