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Lecture 23 1.12.2015

Observe that for $s < t$,

$$B_t^2 = [B_s + (B_t - B_s)]^2 = B_s^2 + 2B_s(B_t - B_s) + (B_t - B_s)^2,$$

$$E[B_t^2|\mathcal{F}_s] = B_s^2 + 2B_s E[(B_t - B_s)|\mathcal{F}_s] + E[(B_t - B_s)^2|\mathcal{F}_s] = B_s^2 + 0 + (t - s) :$$
$$E[B_t^2 - t|\mathcal{F}_s] = B_s^2 - s :$$

$B_t^2 - t$ is a martingale.
Quadratic Variation (QV).

The theory above extends to *continuous* martingales (bounded continuous martingales in general, but we work on a finite time-interval $[0, T]$, so continuity implies boundedness). We quote (for proof, see e.g. [RY], IV.1):

Theorem. A continuous martingale M is of finite quadratic variation $\langle M \rangle$, and $\langle M \rangle$ is the unique continuous increasing adapted process vanishing at zero with $M^2 - \langle M \rangle$ a martingale.

Corollary. A continuous martingale M has infinite variation.

Quadratic Covariation. We write $\langle M, M \rangle$ for $\langle M \rangle$, and extend $\langle\ \rangle$ to a bilinear form $\langle ., . \rangle$ with two different arguments by the *polarization identity*:

$$\langle M, N \rangle := \frac{1}{4}(\langle M + N, M + N \rangle - \langle M - N, M - N \rangle).$$

If N is of *finite* variation, $M \pm N$ has the same QV as M, so $\langle M, N \rangle = 0$.

Itô's Lemma. We discuss Itô's Lemma in more detail in §6 below; we pause here to give the link with quadratic variation and covariation. We quote: if $f(t, x_1, \cdots, x_d)$ is C^1 in its zeroth (time) argument t and C^2 in its remaining d space arguments x_i, and $M = (M^1, \cdots, M^d)$ is a continuous vector martingale, then (writing f_i, f_{ij} for the first partial derivatives of f with respect to its ith argument and the second partial derivatives with respect to the ith and jth arguments) $f(M_t)$ has stochastic differential

$$df(M_t) = f_0(M)dt + \Sigma_{i=1}^d f_i(M_t)dM_t^i + \frac{1}{2}\Sigma_{i,j=1}^d f_{ij}(M_t)d\langle M^i, M^j \rangle_t.$$

Integration by Parts. If $f(t, x_1, x_2) = x_1 x_2$, we obtain

$$d(MN)_t = N dM_t + M dN_t + \frac{1}{2}\langle M, N\rangle_t.$$

Similarly for stochastic integrals (defined below): if $Z_i := \int H_i dM_i$ $(i = 1, 2)$, $d\langle Z_1, Z_2\rangle = H_1 H_2 d\langle M_1, M_2\rangle$.
Note. The integration-by-parts formula – a special case of Itô's Lemma, as above – is in fact *equivalent* to Itô's Lemma: either can be used to derive the other. Rogers & Williams [RW1, IV.32.4] describe the integration-by-parts formula/Itô's Lemma as 'the cornerstone of stochastic calculus'.
Fractals Everywhere.

As we saw, a Brownian path is a *fractal* – a *self-similar* object. So too is its zero-set Z. Fractals were studied, named and popularised by the French mathematician Benôit B. Mandelbrot (1924-2010). See his books, and Michael F. Barnsley: *Fractals everywhere.* Academic Press, 1988.

Fractals *look the same at all scales* – diametrically opposite to the familiar functions of Calculus. In Differential Calculus, a differentiable function has a tangent; this means that locally, its graph *looks straight*; similarly in Integral Calculus. While most continuous functions we encounter are differentiable, at least piecewise (i.e., except for 'kinks'), there is a sense in which the typical, or generic, continuous function is *nowhere differentiable.* Thus Brownian paths may look pathological at first sight – but in fact they are typical!
Hedging in continuous time.

Imagine hedging an option in continuous time. In discrete time, this involves repeatedly rebalancing our portfolio between cash and stock; in continuous time, this has to be done continuously. The relevant stochastic processes (Ch. VI) are *geometric Brownian motion (GBM)*, relatives of BM, which, like BM, have *infinite variation* (finite QV). This makes the rebalancing problematic – indeed, impossible in these terms. Analogy: a cyclist has to rebalance continuously, but does so smoothly, not with infinite variation! Or, think of continuous-time control of a manned space-craft (Kalman filter). In practice, hedging has to be done discretely (as in Ch. IV). Or, we can use price processes with *jumps* – finite variation, but now the markets are incomplete.

§5. Stochastic Integrals (Itô Calculus)

Stochastic integration was introduced by K. ITÔ in 1944, hence its name

Itô calculus. It gives a meaning to $\int_0^t XdY = \int_0^t X_s(\omega)dY_s(\omega)$, for suitable stochastic processes X and Y, the *integrand* and the *integrator*. We shall confine our attention here to the basic case with integrator Brownian motion: $Y = B$. Much greater generality is possible: for Y a continuous martingale, see [KS] or [RY]; for a systematic general treatment, see
MEYER, P.-A. (1976): Un cours sur les intégrales stochastiques. *Séminaire de Probabilités X: Lecture Notes on Math.* **511**, 245-400, Springer.

The first thing to note is that stochastic integrals with respect to Brownian motion, *if* they exist, must be *quite different* from the measure-theoretic integral of Ch. II.2. For, the Lebesgue-Stieltjes integrals described there have as integrators the difference of two monotone (increasing) functions (by Jordan's theorem), which are locally of *finite (bounded) variation, FV*. But we know from §4 that Brownian motion is of *infinite (unbounded)* variation on every interval. So Lebesgue-Stieltjes and Itô integrals must be fundamentally different.

In view of the above, it is quite surprising that Itô integrals can be defined at all. But if we take for granted Itô's fundamental insight that they *can* be, it is obvious how to begin and clear enough how to proceed. We begin with the simplest possible integrands X, and extend successively much as we extended the measure-theoretic integral of Ch. II.

1. *Indicators.*
If $X_t(\omega) = I_{[a,b]}(t)$, there is exactly one plausible way to define $\int XdB$:

$$\int_0^t XdB, \quad \text{or} \quad \int_0^t X_s(\omega)dB_s(\omega), := \begin{cases} 0 & \text{if } t \le a, \\ B_t - B_a & \text{if } a \le t \le b, \\ B_b - B_a & \text{if } t \ge b. \end{cases}$$

2. *Simple functions.* Extend by linearity: if X is a linear combination of indicators, $X = \Sigma c_i I_{[a_i,b_i]}$, we should define

$$\int_0^t XdB := \Sigma c_i \int_0^t I_{[a_i,b_i]}dB.$$

Already one wonders how to extend this from constants c_i to suitable random variables, and one seeks to simplify the obvious but clumsy three-line expressions above. It turns out that finite sums are not essential: one can have infinite sums, but now we take the c_i uniformly bounded.

We begin again, calling X *simple* if there is an infinite sequence

$$0 = t_0 < t_1 < \cdots < t_n < \cdots \to \infty$$

and uniformly bounded $\mathcal{F}_{t_n}$-measurable random variables ξ_n ($|\xi_n| \leq C$ for all n and ω, for some C) if $X_t(\omega)$ can be written in the form

$$X_t(\omega) = \xi_0(\omega)I_{\{0\}}(t) + \Sigma_{i=0}^{\infty}\xi_i(\omega)I_{(t_i,t_{i+1}]}(t) \qquad (0 \leq t < \infty, \omega \in \Omega).$$

The only definition of $\int_0^t XdB$ that agrees with the above for finite sums is, if n is the unique integer with $t_n \leq t < t_{n+1}$,

$$\begin{aligned} I_t(X) := \int_0^t XdB &= \Sigma_0^{n-1}\xi_i(B(t_{i+1}) - B(t_i)) + \xi_n(B(t) - B(t_n)) \\ &= \Sigma_0^{\infty}\xi_i(B(t \wedge t_{i+1}) - B(t \wedge t_i)) \qquad (0 \leq t < \infty). \end{aligned}$$

We note here some properties of the stochastic integral defined so far:
A. $I_0(X) = 0 \qquad P - a.s.$
B. *Linearity.* $I_t(aX + bY) = aI_t(X) + bI_t(Y)$.
Proof. Linear combinations of simple functions are simple.
C. $E[I_t(X)|\mathcal{F}_s] = I_s(X) \qquad P - a.s. \qquad (0 \leq s < t < \infty):$
$I_t(X) = \int_0^t XdB$ is a *continuous martingale.*
Proof. There are two cases to consider.
(i) Both s and t belong to the same interval $[t_n, t_{n+1})$. Then

$$I_t(X) = I_s(X) + \xi_n(B(t) - B(s)).$$

But ξ_n is $\mathcal{F}_{t_n}$-measurable, so $\mathcal{F}_s$-measurable ($t_n \leq s$), so independent of $B(t) - B(s)$ (independent increments property of B). So

$$E[I_t(X)|\mathcal{F}_s] = I_s(X) + \xi_n E[B(t) - B(s)|\mathcal{F}_s] = I_s(X).$$

(ii) $s < t$ and t belong to different intervals: $s \in [t_m, t_{m+1})$ for $m < n$. Then

$$\begin{aligned} E[I_t(x)|\mathcal{F}_s] &= E(E[I_t(X)|\mathcal{F}_{t_n}]|\mathcal{F}_s) \qquad \text{(iterated conditional expectations)} \\ &= E(I_{t_n}(X)|\mathcal{F}_s), \end{aligned}$$

since ξ_n $\mathcal{F}_{t_n}$-measurable and independent increments of B give

$$E[\xi_n(B(t) - B(t_n))|\mathcal{F}_{t_n}] = \xi_n E[B(t) - B(t_n)|\mathcal{F}_{t_n}] = \xi_n.0 = 0.$$

Continuing in this way, we can reduce successively to t_{m+1}:

$$E[I_t(X)|\mathcal{F}_s] = E[I_{t_m}(X)|\mathcal{F}_s].$$

But $I_{t_m}(X) = I_s(X) + \xi_m(B(s) - B(t_m))$; taking $E[.|\mathcal{F}_s]$ the second term gives zero as above, giving the result. //