m3a22l21tex

Lecture 21 27.10.2015

Filtrations and insider trading (ctd).

Again necessarily, many people are involved in major business projects and decisions (an important example: mergers and acquisitions, or M&A) involving publicly quoted companies. Frequently, this involves price-sensitive information. People in this position are – rightly – prohibited by law from profiting by it directly, by trading on their own account, in publicly quoted stocks but using private information. This is rightly regarded as theft at the expense of the investing public.[1] Instead, those involved in M&A etc. should seek to benefit legitimately (and indirectly) – enhanced career prospects, commission or fees, bonuses etc.

The regulatory authorities (Securities and Exchange Commission – SEC – in US; Financial Conduct Authority (FCA) and Prudential Regulation Authority (PRA, part of the Bank of England (BoE) in UK) monitor all trading electronically. Their software alerts them to patterns of suspicious trades. The software design (necessarily secret, in view of its value to criminals) involves all the necessary elements of Mathematical Finance in exaggerated form: economic and financial insight, plus: mathematics; statistics (especially pattern recognition, data mining and machine learning); numerics and computation.

§2. Classes of Processes.

1. *Martingales.*

The martingale property in continuous time is just that suggested by the discrete-time case:

$$E[X_t|\mathcal{F}_s] = X_s \qquad (s < t),$$

and similarly for submartingales and supermartingales. There are regularization results, under which one can take X_t right-continuous in t. Among the contrasts with the discrete case, we mention that the Doob-Meyer decomposition, easy in discrete time (III.8), is a deep result in continuous time. For background, see e.g.

MEYER, P.-A. (1966): *Probabilities and potentials.* Blaisdell

- and subsequent work by Meyer and the French school (Dellacherie & Meyer, *Probabilités et potentiel*, I-V, etc.

[1] The plot of the film *Wall Street* revolves round such a case, and is based on real life – recommended!

2. Gaussian Processes.

Recall the multivariate normal distribution $N(\mu, \mathbf{\Sigma})$ in n dimensions. If $\mu \in \mathbb{R}^n$, $\mathbf{\Sigma}$ is a non-negative definite $n \times n$ matrix, $\mathbf{X}$ has distribution $N(\mu, \mathbf{\Sigma})$ if it has characteristic function

$$\phi_{\mathbf{X}}(\mathbf{t}) := E \exp\{i\mathbf{t}^T.\mathbf{X}\} = \exp\{i\mathbf{t}^T.\mu - \frac{1}{2}\mathbf{t}^T\mathbf{\Sigma}\mathbf{t}\} \qquad (\mathbf{t} \in \mathbb{R}^n).$$

If further $\mathbf{\Sigma}$ is positive definite (so non-singular), $\mathbf{X}$ has density (*Edgeworth's Theorem* of 1893: F. Y. Edgeworth (1845-1926), English statistician)

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{1}{2}n}|\Sigma|^{\frac{1}{2}}} \exp\{-\frac{1}{2}(\mathbf{x} - \mu)^T\mathbf{\Sigma}^{-1}(\mathbf{x} - \mu)\}.$$

A process $X = (X_t)_{t \geq 0}$ is *Gaussian* if all its finite-dimensional distributions are Gaussian. Such a process can be specified by:
(i) a measurable function $\mu = \mu(t)$ with $EX_t = \mu(t)$,
(ii) a non-negative definite function $\sigma(s, t)$ with

$$\sigma(s, t) = cov(X_s, X_t).$$

Gaussian processes have many interesting properties. Among these, we quote *Belayev's dichotomy*: with probability one, the paths of a Gaussian process are either continuous, or extremely pathological: for example, unbounded above and below on any time-interval, however short. Naturally, we shall confine attention in this course to continuous Gaussian processes.

3. Markov Processes.

X is *Markov* if for each t, each $A \in \sigma(X_s : s > t)$ (the 'future') and $B \in \sigma(X_s : s < t)$ (the 'past'),

$$P(A|X_t, B) = P(A|X_t).$$

That is, if you know where you are (at time t), how you got there doesn't matter so far as predicting the future is concerned – equivalently, past and future are conditionally independent given the present.

The same definition applied to Markov processes in discrete time.

X is said to be *strong Markov* if the above holds with the *fixed* time t replaced by a *stopping time* T (a random variable). This is a real restriction of the Markov property in continuous time (though not in discrete time) – another instance of the difference between the two.

4. Diffusions.

A *diffusion* is a path-continuous strong-Markov process such that for each time t and state x the following limits exist:

$$\mu(t,x) := \lim_{h\downarrow 0} \frac{1}{h} E[(X_{t+h} - X_t)|X_t = x],$$

$$\sigma^2(t,x) := \lim_{h\downarrow 0} \frac{1}{h} E[(X_{t+h} - X_t)^2|X_t = x].$$

Then $\mu(t,x)$ is called the *drift*, $\sigma^2(t,x)$ the *diffusion coefficient.* Then $p(t,x,y)$, the density of transitions from x to y in time t, satisfies the parabolic PDE

$$Lp = \partial p/\partial t, \qquad L := \frac{1}{2}\sigma^2 D^2 + \mu(x)D, \qquad D := \partial/\partial x.$$

Brownian motion (below) is the case $\sigma = 1$, $\mu = 0$, and gives the *heat equation* ($L = \frac{1}{2}D^2$ in one dimension, half the *Laplacian* Δ in higher dimensions).

§3. Brownian Motion.

The Scottish botanist Robert Brown observed pollen particles in suspension under a microscope in 1828 and 1829 (though this had been observed before),[2] and observed that they were in constant irregular motion.

In 1900 L. Bachelier considered Brownian motion a possible model for stock-market prices:
BACHELIER, L. (1900): Théorie de la spéculation. *Ann. Sci. Ecole Normale Supérieure* **17**, 21-86
– the first time Brownian motion had been used to model financial or economic phenomena, and before a mathematical theory had been developed.

In 1905 Albert Einstein considered Brownian motion as a model of particles in suspension, and used it to estimate *Avogadro's number* ($N \sim 6\times 10^{23}$), based on the diffusion coefficient D in the *Einstein relation*

$$var X_t = Dt \qquad (t > 0).$$

In 1923 Norbert WIENER defined and constructed Brownian motion rigorously for the first time. The resulting stochastic process is often called the *Wiener process* in his honour, and its probability measure (on path-space) is

[2]The Roman author Lucretius observed this phenomenon in the gaseous phase – dust particles dancing in sunbeams – in antiquity: *De rerum naturae*, c. 50 BC.

called *Wiener measure.*

We define *standard Brownian motion* on $\mathbb{R}$, BM or $BM(\mathbb{R})$, to be a stochastic process $X = (X_t)_{t\geq 0}$ such that
1. $X_0 = 0$,
2. X has *independent increments*: $X_{t+u} - X_t$ is independent of $\sigma(X_s : s \leq t)$ for $u \geq 0$,
3. X has *stationary increments*: the law of $X_{t+u} - X_t$ depends only on u,
4. X has *Gaussian increments*: $X_{t+u} - X_t$ is normally distributed with mean 0 and variance u,

$$X_{t+u} - X_t \sim N(0, u),$$

5. X has *continuous paths*: X_t is a continuous function of t, i.e. $t \mapsto X_t$ is continuous in t.

For time t in a finite interval – $[0, 1]$, say – we can use the following filtered space: (i) $\Omega = C[0, 1]$, the space of all continuous functions on $[0, 1]$; (ii) the points $\omega \in \Omega$ are thus random functions, and we use the coordinate mappings: X_t, or $X_t(\omega)$, $= \omega_t$; (iii) the filtration is given by $\mathcal{F}_t := \sigma(X_s : 0 \leq s \leq t)$, $\mathcal{F} := \mathcal{F}_1$; (iv) P is the measure on $(\Omega, \mathcal{F})$ with finite-dimensional distributions specified by the restriction that the increments $X_{t+u} - X_t$ are stationary independent Gaussian $N(0, u)$.

Theorem (WIENER, 1923). Brownian motion exists.

The best way to prove this is by construction, and one that reveals some properties. The proof that follows is originally due to Paley, Wiener and Zygmund (1933) and Lévy (1948), but is re-written in the modern language of *wavelet* expansions. We omit details; for these, see e.g. [BK] 5.3.1, or SP L20-22. The Haar system $(H_n) = (H_n(.))$ is a complete orthonormal system (cons) of functions in $L^2[0, 1]$. The Schauder System Δ_n) is obtained by integrating the Haar system. Consider the triangular function (or 'tent function')

$$\Delta(t) := 2t \quad \text{on} \quad [0, \frac{1}{2}), \qquad 2(1-t) \quad \text{on } [\frac{1}{2}, 1], \qquad 0 \quad \text{else.}$$

With $\Delta_0(t) := t$, $\Delta_1(t) := \Delta(t)$, define the nth *Schauder function* Δ_n by

$$\Delta_n(t) := \Delta(2^j t - k) \qquad (n = 2^j + k \geq 1).$$

Note that Δ_n has support $[k/2^j, (k+1)/2^j]$ (so is 'localized' on this dyadic interval, which is small for n, j large).