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Lecture 15 13.10.2015

The converse is true, but harder, and needs a preparatory result – which is interesting and important in its own right.

Separating Hyperplane Theorem (SHT).

In a vector space V, a *hyperplane* is a translate of a (vector) subspace U of codimension 1 – that is, U and some one-dimensional subspace, say $\mathbb{R}$, together span V: V is the direct sum $V = U \oplus \mathbb{R}$ (e.g., $\mathbb{R}^3 = \mathbb{R}^2 \oplus \mathbb{R}$). Then

$$H = [f, \alpha] := \{x : f(x) = \alpha\}$$

for some α and linear functional f. In the finite-dimensional case, of dimension n, say, one can think of $f(x)$ as an inner product,

$$f(x) = f.x = f_1 x_1 + \ldots + f_n x_n.$$

The hyperplane $H = [f, \alpha]$ *separates* sets $A, B \subset V$ if

$$f(x) \geq \alpha \qquad \forall\, x \in A, \qquad f(x) \leq \alpha \qquad \forall\, x \in B$$

(or the same inequalities with A, B, or $\geq, \leq$, interchanged).

Call a set A in a vector space V *convex* if

$$x, y \in A, \quad 0 \leq \lambda \leq 1 \quad \Rightarrow \quad \lambda x + (1 - \lambda) y \in A$$

– that is, A contains the line-segment joining any pair of its points.

We can now state (without proof) the SHT (see e,g, [BK] App. C).

SHT. Any two non-empty disjoint convex sets in a vector space can be separated by a hyperplane.

A *cone* is a subset of a vector space closed under vector addition and multiplication by *positive* constants (so: like a vector subspace, but with a sign-restriction in scalar multiplication).

We turn now to the proof of the converse.

Proof of the converse (not examinable). $\Rightarrow$: Write Γ for the cone of strictly positive random variables. Viability (NA) says that for any admissible strategy H,

$$V_0(H) = 0 \qquad \Rightarrow \qquad \tilde{V}_N(H) \notin \Gamma. \qquad (*)$$

To any admissible process $(H_n^1, \cdots, H_n^d)$, we associate its discounted cumulative *gain* process

$$\tilde{G}_n(H) := \Sigma_1^n (H_j^1 \Delta \tilde{S}_j^1 + \cdots + H_j^d \Delta \tilde{S}_j^d).$$

By the Proposition, we can extend $(H_1, \cdots, H_d)$ to a unique predictable process (H_n^0) such that the strategy $H = ((H_n^0, H_n^1, \cdots, H_n^d))$ is self-financing with initial value zero. By NA, $\tilde{G}_N(H) = 0$ – that is, $\tilde{G}_N(H) \notin \Gamma$.

We now form the set $\mathcal{V}$ of random variables $\tilde{G}_N(H)$, with $H = (H^1, \cdots, H^d)$ a previsible process. This is a vector subspace of the vector space $\mathbb{R}^\Omega$ of random variables on Ω, by linearity of the gain process $G(H)$ in H. By $(*)$, this subspace $\mathcal{V}$ does not meet Γ. So $\mathcal{V}$ does not meet the subset

$$K := \{X \in \Gamma : \Sigma_\omega X(\omega) = 1\}.$$

Now K is a convex set not meeting the origin. By the Separating Hyperplane Theorem, there is a vector $\lambda = (\lambda(\omega) : \omega \in \Omega)$ such that for all $X \in K$

$$\lambda.X := \Sigma_\omega \lambda(\omega) X(\omega) > 0, \tag{1}$$

but for all $\tilde{G}_N(H)$ in $\mathcal{V}$,

$$\lambda.\tilde{G}_N(H) = \Sigma_\omega \lambda(\omega) \tilde{G}_N(H)(\omega) = 0. \tag{2}$$

Choosing each $\omega \in \Omega$ successively and taking X to be 1 on this ω and zero elsewhere, (1) tells us that each $\lambda(\omega) > 0$. So

$$P^*(\{\omega\}) := \lambda(\omega)/(\Sigma_{\omega' \in \Omega} \lambda(\omega'))$$

defines a probability measure equivalent to P (no non-empty null sets). With E^* as P^*-expectation, (2) says that

$$E^*[\tilde{G}_N(H)] = 0 : \qquad E^*[\Sigma_1^N H_j.\Delta \tilde{S}_j] = 0.$$

In particular, choosing for each i to hold only stock i,

$$E^*[\Sigma_1^N H_j^i \Delta \tilde{S}_j^i] = 0 \qquad (i = 1, \cdots, d).$$

By the Martingale Transform Lemma, this says that the discounted price processes $(\tilde{S}_n^i)$ are P^*-martingales. //

§3. Complete Markets: Uniqueness of EMMs.

A contingent claim (option, etc.) can be defined by its *payoff* function, h say, which should be non-negative (options confer rights, not obligations, so negative values are avoided by not exercising the option), and $\mathcal{F}_N$-measurable (so that we know how to evaluate h at the terminal time N).

Definition. A contingent claim defined by the payoff function h is *attainable* if there is an admissible strategy worth (i.e., replicating) h at time N. A market is *complete* if every contingent claim is attainable.

Theorem (Completeness Theorem: complete iff EMM unique). A viable market is complete iff there exists a unique probability measure P^* equivalent to P under which discounted asset prices are martingales – that is, iff equivalent martingale measures are unique.

Proof. $\Rightarrow$: Assume viability and completeness. Then for any $\mathcal{F}_N$-measurable random variable $h \geq 0$, there exists an admissible (so SF) strategy H replicating h: $h = V_N(H)$. As H is SF, by §1

$$h/S_N^0 = \tilde{V}_N(H) = V_0(H) + \Sigma_1^N H_j.\Delta\tilde{S}_j.$$

We know by the Theorem of §2 that an equivalent martingale measure P^* exists; we have to prove uniqueness. So, let P_1, P_2 be two such equivalent martingale measures. For $i = 1, 2$, $(\tilde{V}_n(H))_{n=0}^N$ is a P_i-martingale. So,

$$E_i[\tilde{V}_N(H)] = E_i[V_0(H)] = V_0(H),$$

since the value at time zero is non-random ($\mathcal{F}_0 = \{\emptyset, \Omega\}$). So

$$E_1[h/S_N^0] = E_2[h/S_N^0].$$

Since h is arbitrary, E_1, E_2 have to agree on integrating all non-negative integrands. Taking negatives and using linearity: they have to agree on non-positive integrands also. Splitting an arbitrary integrand into its positive and negative parts: they have to agree on all integrands. Now E_i is expectation (i.e., integration) with respect to the measure P_i, and measures that agree on integrating all integrands must coincide. So $P_1 = P_2$. //

Before proving the converse, we prove a lemma. Recall that an admissible strategy is a SF strategy with all values non-negative. The Lemma shows that the non-negativity of contingent claims extends to all values of any SF strategy replicating it – in other words, this gives equivalence of admissible and SF replicating strategies. [SF: isolated from external wealth; admissible:

actually worth something. These sound similar; the Lemma shows they are the same here. So we only need one term; we use SF as it is shorter, but w.l.o.g. this means admissible also.]

Lemma. In a viable market, any attainable h (i.e., any h that can be replicated by a SF strategy H) can also be replicated by an admissible strategy H.

Proof. If H is SF and P^* is an equivalent martingale measure under which discounted prices $\tilde{S}$ are P^*-martingales (such P^* exist by viability and the Theorem of §2), $\tilde{V}_n(H)$ is also a P^*-martingale, being the martingale transform of $\tilde{S}$ by H (see §1). So

$$\tilde{V}_n(H) = E^*[\tilde{V}_N(H)|\mathcal{F}_n] \qquad (n = 0, 1, \cdots, N).$$

If H replicates h, $V_N(H) = h \geq 0$, so discounting, $\tilde{V}_N(H) \geq 0$, so the above equation gives $\tilde{V}_n(H) \geq 0$ for each n. Thus *all* the values at each time n are non-negative – not just the final value at time N – so H is admissible. //

Proof of the Theorem (continued). $\Leftarrow$ (*not examinable)*: Assume the market is viable but incomplete: then there exists a non-attainable $h \geq 0$. By the Proposition of §1, we may confine attention to the risky assets $S^1, \cdots, S^d$, as these suffice to tell us how to handle the bank account S^0.

Call $\tilde{\mathcal{V}}$ the set of random variables of the form

$$U_0 + \Sigma_1^N H_n . \Delta \tilde{S}_n$$

with U_0 $\mathcal{F}_0$-measurable (i.e. deterministic) and $((H_n^1, \cdots, H_n^d))_{n=0}^N$ predictable; this is a vector space. (Here $(H^1, \ldots, H^d)$ extends to $H := (H^0, H^1, \ldots, H^d)$, by the Proposition of §1, and H can be any strategy here.) Then as h is not attainable, the discounted value h/S_N^0 does not belong to $\tilde{\mathcal{V}}$, so $\tilde{\mathcal{V}}$ is a *proper* subspace of the vector space $\mathbb{R}^\Omega$ of all random variables on Ω. Let P^* be a probability measure equivalent to P under which discounted prices are martingales (such P^* exist by viability, by the Theorem of §2). Define the scalar product

$$(X, Y) \to E^*[XY]$$

on random variables on Ω. Since $\tilde{\mathcal{V}}$ is a proper subspace, by Gram-Schmidt orthogonalisation there exists a non-zero random variable X orthogonal to $\tilde{\mathcal{V}}$. That is,

$$E^*[X] = 0.$$