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Lecture 14 10.10.2015

Chapter IV. MATHEMATICAL FINANCE IN DISCRETE TIME.

We follow [BK], Ch. 4 (or see the other sources cited in L0).

§1. The Model.

It suffices, to illustrate the ideas, to work with a *finite* probability space $(\Omega, \mathcal{F}, \mathcal{P})$, with a finite number $|\Omega|$ of points ω, each with positive probability: $P(\{\omega\}) > 0$. We will use a finite time-horizon N, which will correspond to the expiry date of the options.

As before, we use a filtration $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_N$: we may (and shall) take $\mathcal{F}_0 = \{\emptyset, \Omega\}$, the trivial σ-field, $\mathcal{F}_N = \mathcal{F} = \mathcal{P}(\Omega)$ (here $\mathcal{P}(\Omega)$ is the *power-set* of Ω, the class of all $2^{|\Omega|}$ subsets of Ω: we need every possible subset, as they all (apart from the empty set) carry positive probability.

The financial market contains $d+1$ financial assets: a riskless asset (bank account) labelled 0, and d risky assets (stocks, say) labelled 1 to d. The prices of the assets at time n are random variables, $S_n^0, S_n^1, \cdots, S_n^d$ say [note that we use superscripts here as labels, *not* powers, and suppress ω for brevity], non-negative and $\mathcal{F}_n$-measurable [at time n, we know the prices S_n^i].

We take $S_0^0 = 1$ (that is, we reckon in units of our initial bank holding). We assume for convenience a constant interest rate $r > 0$ in the bank, so 1 unit in the bank at time 0 grows to $(1+r)^n$ at time n. So $1/(1+r)^n$ is the *discount factor* at time n.

Definition. A *trading strategy* H is a vector stochastic process $H = (H_n)_{n=0}^N = ((H_n^0, H_n^1, \cdots, H_n^d))_{n=0}^N$ which is *predictable* (or *previsible*): each H_n^i is $\mathcal{F}_{n-1}$-measurable for $n \geq 1$.

Here H_n^i denotes the number of shares of asset i held in the portfolio at time n – to be determined on the basis of information available *before* time n; the vector $H_n = (H_n^0, H_n^1, \cdots, H_n^d)$ is the *portfolio* at time n. Writing $S_n = (S_n^0, S_n^1, \cdots, S_n^d)$ for the vector of prices at time n, the *value* of the portfolio at time n is the scalar product

$$V_n(H) = H_n.S_n := \Sigma_{i=0}^d H_n^i S_n^i.$$

The *discounted value* is

$$\tilde{V}_n(H) = \beta_n(H_n.S_n) = H_n.\tilde{S}_n,$$

where $\beta_n := 1/S_n^0$ and $\tilde{S}_n = (1, \beta_n S_n^1, \cdots, \beta_n S_n^d)$ is the vector of discounted prices.
Note. The *previsibility* of H reflects that there is no *insider trading*.

Definition. The strategy H is *self-financing (SF)*, $H \in SF$, if

$$H_n.S_n = H_{n+1}.S_n \qquad (n = 0, 1, \cdots, N-1).$$

Interpretation. When new prices S_n are quoted at time n, the investor adjusts his portfolio from H_n to H_{n+1}, without bringing in or consuming any wealth.

$$\begin{aligned} V_{n+1}(H) - V_n(H) &= H_{n+1}.S_{n+1} - H_n.S_n \\ &= H_{n+1}.(S_{n+1} - S_n) + (H_{n+1}.S_n - H_n.S_n). \end{aligned}$$

For a SF strategy, the second term on the right is zero. Then the LHS, the net increase in the value of the portfolio, is shown as due only to the price changes $S_{n+1} - S_n$. So for $H \in SF$,

$$V_n(H) - V_{n-1}(H) = H_n(S_n - S_{n-1}),$$

$$\Delta V_n(H) = H_n.\Delta S_n, \qquad V_n(H) = V_0(H) + \Sigma_1^n H_j.\Delta S_j$$

and $V_n(H)$ is the *martingale transform* of S by H (III.6). Similarly with discounting:

$$\Delta \tilde{V}_n(H) = H_n.\Delta \tilde{S}_n, \qquad \tilde{V}_n(H) = V_0(H) + \Sigma_1^n H_j.\Delta \tilde{S}_j$$

$(\Delta \tilde{S}_n := \tilde{S}_n - \tilde{S}_{n-1} = \beta_n S_n - \beta_{n-1} S_{n-1})$.

As in I, we are allowed to borrow (so S_n^0 may be negative) and sell short (so S_n^i may be negative for $i = 1, \cdots, d$). So it is hardly surprising that if we decide what to do about the risky assets, the bank account will take care of itself, in the following sense.

Proposition. If $((H_n^1, \cdots, H_n^d))_{n=0}^N$ is predictable and V_0 is $\mathcal{F}_0$-measurable, there is a unique predictable process $(H_n^0)_{n=0}^N$ such that $H = (H^0, H^1, \cdots, H^d)$ is SF with initial value V_0.

Proof. If H is SF, then as above

$$\tilde{V}_n(H) = H_n.\tilde{S}_n = H_n^0 + H_n^1 \tilde{S}_n^1 + \cdots + H_n^d \tilde{S}_n^d,$$

while as $\tilde{V}_n = H.\tilde{S}_n$,

$$\tilde{V}_n(H) = V_0 + \Sigma_1^n (H_j^1 \Delta \tilde{S}_j^1 + \cdots + H_j^d \Delta \tilde{S}_j^d)$$

($\tilde{S}_n = (1, \beta_n S_n^1, \cdots, \beta_n S_n^d)$, so $\tilde{S}_n^0 \equiv 1$, $\Delta \tilde{S}_n^0 = 0$). Equate these:

$$H_n^0 = V_0 + \Sigma_1^n (H_j^1 \Delta \tilde{S}_j^1 + \cdots + H_j^d \Delta \tilde{S}_j^d) - (H_n^1 \tilde{S}_n^1 + \cdots + H_n^d \tilde{S}_n^d),$$

which defines H_n^0 uniquely. The terms in $\tilde{S}_n^i$ are $H_n^i \Delta \tilde{S}_n^i - H_n^i \tilde{S}_n^i = -H_n^i \tilde{S}_{n-1}^i$, which is $\mathcal{F}_{n-1}$-measurable. So

$$H_n^0 = V_0 + \Sigma_1^{n-1} (H_j^1 \Delta \tilde{S}_j^1 + \cdots + H_j^d \Delta \tilde{S}_j^d) - (H_n^1 S_{n-1}^1 + \cdots + H_n^d \tilde{S}_{n-1}^d),$$

where as $H^1, \cdots, H^d$ are predictable, all terms on the RHS are $\mathcal{F}_{n-1}$-measurable, so H^0 is predictable. //

Numéraire. What units do we reckon value in? All that is really necessary is that our chosen unit of account should always be *positive* (as we then reckon our holdings by dividing by it, and one cannot divide by zero). Common choices are pounds sterling (UK), dollars (US), euros etc. Gold is also possible (now priced in sterling etc. – but the pound sterling represented an amount of gold, till the UK 'went off the gold standard'). By contrast, risky stocks *can* have value 0 (if the company goes bankrupt). We call such an always-positive asset, used to reckon values in, a *numéraire*.

Of course, one has to be able to change numéraire – e.g. when going from UK to the US or eurozone. As one would expect, this changes nothing important. In particular, we quote (*numéraire invariance theorem* – see e.g. [BK] Prop. 4.1.1) that the set SF of self-financing strategies is invariant under change of numéraire.

Note. 1. This alerts us to what is meant by 'risky'. To the owner of a goldmine, sterling is risky. The danger is not that the UK government might go bankrupt, but that sterling might depreciate against the dollar, or euro, etc.
2. With this understood, we shall feel free to refer to our numéraire as 'bank account'. The point is that we don't trade in it (why would a goldmine owner trade in gold?); it is the other – 'risky' – assets that we trade in.

§2. Viability (NA): Existence of Equivalent Martingale Measures.

Although we are allowed to borrow (from the bank), and sell (stocks) short, we are – naturally – required to stay solvent (recall that trading while insolvent is an offence under the Companies Act!).

Definition. A strategy H is *admissible* if it is self-financing (SF), and $V_n(H) \geq 0$ for each time $n = 0, 1, \cdots, N$.

Recall that arbitrage is riskless profit – making 'something out of nothing'. Formally:

Definition. An *arbitrage strategy* is an admissible strategy with zero initial value and positive probability of a positive final value.
Definition. A market is *viable* if no arbitrage is possible, i.e. if the market is arbitrage-free (no-arbitrage, NA).

This leads to the first of two fundamental results:

Theorem (No-Arbitrage Theorem: NA iff EMMs exist). The market is viable (is arbitrage-free, is NA) iff there exists a probability measure P^* equivalent to P (i.e., having the same null sets) under which the discounted asset prices are P^*-martingales – that is, iff there exists an equivalent martingale measure (EMM).

Proof. $\Leftarrow$. Assume such a P^* exists. For any self-financing strategy H, we have as before

$$\tilde{V}_n(H) = V_0(H) + \Sigma_1^n H_j.\Delta\tilde{S}_j.$$

By the Martingale Transform Lemma, $\tilde{S}_j$ a (vector) P^*-martingale implies $\tilde{V}_n(H)$ is a P^*-martingale. So the initial and final P^*-expectations are the same: using E^* for P^*-expectation,

$$E^*[\tilde{V}_N(H)] = E^*[\tilde{V}_0(H)].$$

If the strategy is admissible and its initial value – the RHS above – is zero, the LHS $E^*[\tilde{V}_N(H)]$ is zero, but $\tilde{V}_N(H) \geq 0$ (by admissibility). Since each $P(\{\omega\}) > 0$ (by assumption), each $P^*(\{\omega\}) > 0$ (by equivalence). This and $\tilde{V}_N(H) \geq 0$ force $\tilde{V}_N(H) = 0$ (sum of non-negatives can only be 0 if each term is 0). So no arbitrage is possible. //