m3a22l12tex

Lecture 12 6.11.2015

Martingale convergence (continued).

More is true. Call X L_1-bounded if

$$\sup_n E[|X_n|] < \infty.$$

Theorem (Doob). An L_1-bounded supermartingale is a.s. convergent: there exists X_∞ finite such that

$$X_n \to X_\infty \qquad (n \to \infty) \qquad a.s.$$

In particular, we have

Doob's Martingale Convergence Theorem [W, §11.5]. An L_1-bounded martingale converges a.s.

We say that

$$X_n \to X_\infty \text{ in } L_1$$

if

$$E[|X_n - X_\infty|] \to 0 \qquad (n \to \infty).$$

For a class of martingales, one gets convergence in L_1 as well as almost surely [= with probability one]. Such martingales are called *uniformly integrable* (UI) [W], or *regular* [N], or *closed* (see below), They are "the nice ones". Fortunately, they are the ones we need.

The following result is in [N], IV.2, [W], Ch. 14; cf. SP L18-19, SA L6.

Theorem (UI Martingale Convergence Theorem). The following are equivalent for martingales $X = (X_n)$:
(i) X_n converges in L_1,
(ii) X_n is L_1-bounded, and its a.s. limit X_∞ (which exists, by above) satisfies

$$X_n = E[X_\infty|\mathcal{F}_n],$$

(iii) There exists an integrable random variable X with

$$X_n = E[X|\mathcal{F}_n].$$

The random variable X_∞ above serves to "close" the martingale, by giving X_n a value at "$n = \infty$"; then $\{X_n : n = 1, 2, \ldots, \infty\}$ is again a martingale – which we may accordingly call a closed mg. The terms closed, regular and UI are used interchangeably here.

Notice that *all the randomness in a closed mg is in the closing value* X_∞ (so, although a stochastic process is an infinite-dimensional object, the randomness in a closed mg is one-dimensional). As time progresses, more is revealed, by "progressive revelation" – as in (choose your metaphor) a striptease, or the "Day of Judgement" (when all will be revealed).

As we shall see (Risk-Neutral Valuation Formula): closed mgs are vital in mathematical finance, and the closing value corresponds to the payoff of an option.

§5. Martingale Transforms.

Now think of a gambling game, or series of speculative investments, in discrete time. There is no play at time 0; there are plays at times $n = 1, 2, \cdots$, and

$$\Delta X_n := X_n - X_{n-1}$$

represents our net winnings per unit stake at play n. Thus if X_n is a martingale, the game is 'fair on average'.

Call a process $C = (C_n)_{n=1}^\infty$ *previsible* (or *predictable*) if

$$C_n \text{ is } \mathcal{F}_{n-1} - \text{measurable for all } n \geq 1.$$

Think of C_n as your stake on play n (C_0 is not defined, as there is no play at time 0). Previsibility says that you have to decide how much to stake on play n based on the history *before* time n (i.e., up to and including play $n - 1$). Your winnings on game n are $C_n \Delta X_n = C_n(X_n - X_{n-1})$. Your total (net) winnings up to time n are

$$Y_n = \sum_{k=1}^n C_k \Delta X_k = \sum_{k=1}^n C_k (X_k - X_{k-1}).$$

We write

$$Y = C \bullet X, \qquad Y_n = (C \bullet X)_n, \qquad \Delta Y_n = C_n \Delta X_n$$

($(C \bullet X)_0 = 0$ as $\sum_1^0$ is empty), and call $C \bullet X$ the *martingale transform* of X by C.

Theorem. (i) If C is a bounded non-negative previsible process and X is a supermartingale, $C \bullet X$ is a supermartingale null at zero.
(ii) If C is bounded and previsible and X is a martingale, $C \bullet X$ is a martingale null at zero.

Proof. With $Y = C \bullet X$ as above,

$$E[Y_n - Y_{n-1}|\mathcal{F}_{n-1}] = E[C_n(X_n - X_{n-1})|\mathcal{F}_{n-1}]$$

$$= C_n E[(X_n - X_{n-1})|\mathcal{F}_{n-1}]$$

(as C_n is bounded, so integrable, and $\mathcal{F}_{n-1}$-measurable, so can be taken out)

$$\leq 0$$

in case (i), as $C \geq 0$ and X is a supermartingale,

$$= 0$$

in case (ii), as X is a martingale. //

Interpretation. You can't beat the system!
In the martingale case, previsibility of C means we can't foresee the future (which is realistic and fair). So we expect to gain nothing – as we should.
Note. 1. Martingale transforms were introduced and studied by Donald L. BURKHOLDER (1927 - 2013) in 1966 [*Ann. Math. Statist.* **37**, 1494-1504]. For a textbook account, see e.g. [N], VIII.4.
2. Martingale transforms are the discrete analogues of stochastic integrals. They dominate the mathematical theory of finance in discrete time, just as stochastic integrals dominate the theory in continuous time.
3. In mathematical finance, X plays the role of a price process, C plays the role of our trading strategy, and the mg transform $C \bullet X$ plays the role of our gains (or losses!) from trading.

Proposition (Martingale Transform Lemma). An adapted sequence of real integrable random variables (M_n) is a martingale iff for any bounded previsible sequence (H_n),

$$E[\sum_{r=1}^{n} H_r \Delta M_r] = 0 \qquad (n = 1, 2, \cdots).$$

Proof. If (M_n) is a martingale, X defined by $X_0 = 0$, $X_n = \sum_1^n H_r \Delta M_r \qquad (n \geq 1)$ is the martingale transform $H \bullet M$, so is a martingale.

Conversely, if the condition of the Proposition holds, choose j, and for any $\mathcal{F}_j$-measurable set A write $H_n = 0$ for $n \neq j+1$, $H_{j+1} = I_A$. Then (H_n) is previsible, so the condition of the Proposition, $E[\sum_1^n H_r \Delta M_r] = 0$, becomes

$$E[I_A(M_{j+1} - M_j)] = 0.$$

As this holds for every $A \in \mathcal{F}_j$, the definition of conditional expectation gives

$$E[M_{j+1}|\mathcal{F}_j] = M_j.$$

Since this holds for every j, (M_n) is a martingale. //

§6. Stopping Times and Optional Stopping.

A random variable T taking values in $\{0, 1, 2, \cdots; +\infty\}$ is called a *stopping time* (or *optional time*) if

$$\{T \leq n\} = \{\omega : T(\omega) \leq n\} \in \mathcal{F}_n \qquad \forall n \leq \infty.$$

Equivalently,

$$\{T = n\} \in \mathcal{F}_n \qquad n \leq \infty.$$

Think of T as a time at which you decide to quit a gambling game: whether or not you quit at time n depends only on the history up to and including time n – NOT the future. [Elsewhere, T denotes the expiry time of an option. If we mean T to be a stopping time, we will say so.]

The following important classical theorem is discussed in [W], 10.10.

Theorem (Doob's Optional Stopping Theorem, OST). Let T be a stopping time, $X = (X_n)$ be a supermartingale, and assume that one of the following holds:
(i) T is bounded [$T(\omega) \leq K$ for some constant K and all $\omega \in \Omega$];
(ii) $X = (X_n)$ is bounded [$|X_n(\omega)| \leq K$ for some K and all n, ω];
(iii) $E[T] < \infty$ and $(X_n - X_{n-1})$ is bounded.

Then X_T is integrable, and

$$E[X_T] \leq E[X_0].$$

If here X is a martingale, then

$$E[X_T] = E[X_0].$$