M3A22/M4A22/M5A22 MATHEMATICAL FINANCE: EXAM SOLUTIONS 2014-15

Q1. *Perfect (Frictionless) Markets.* For simplicity, we shall confine ourselves to option pricing in the simplest (idealised) case, of a *perfect*, or *frictionless*, market. This entails various assumptions:

No transaction costs. We assume that there is no financial friction in the form of transaction costs (true only to a first approximation). **[2]**

No taxes. We assume similarly that there are no taxes. We note that a *Tobin tax*, designed partly to damp down excessive volumes of trading and partly to raise money for good causes, has recently been suggested. **[2]**

Same interest rates for borrowing and for lending. This is clearly unrealistic, as banks make their money on the difference. But it is a reasonable first approximation, and simplifies such things as arbitrage arguments. **[2]**

Perfect information. We assume that all market participants have perfect information about the past history of price movements, but have no foreknowledge of price-sensitive information (i.e. no insider trading) – also, no information asymmetry, so all participants equally knowledgeable. **[2]**

No liquidity restrictions. That is, one can buy or sell unlimited quantities of stock at the currently quoted price. However, in a crisis, no-one wants to trade, and liquidity dries up. **[2]**

Economic agents are price takers and not price makers. In practice, this is true for small market participants but not for large ones. Big trades do move markets (price is the level at which supply and demand balance; big trades affect this balance significantly). **[2]**

This restriction emphasizes the difference between Economics and Finance. Much of Economics is concerned with *how prices are arrived at* (supply and demand, etc.). In Finance, we take prices as given. **[2]**

No restriction on order size; no delay in executing orders. In practice, executing small orders is uneconomic, so there are size limitations. Also, orders are dealt with in job lots, for efficiency. Delays do occur in executing orders, particularly large ones. **[2]**

No credit risk. Perfect markets assume that all market participants are willing and able to honour their commitments. This ignores the risk of bankruptcy, etc. (necessary, as limited liability is needed to give ordinary market participants the confidence to engage in trade). **[2]**

Other risks, e.g.: fraud; human error; insider trading; etc. **[2]**

[Seen – lectures]

Q2. (i) *Proof of Merton's theorem.* Consider the following two portfolios:
I: one American call option plus cash Ke^{-rT}; II: one share.
The value of the cash in I is K at time T, $Ke^{-r(T-t)}$ at time t. If the call option is exercised early at $t < T$, the value of Portfolio I is then $S_t - K$ from the call, $Ke^{-r(T-t)}$ from the cash, total

$$S_t - K + Ke^{-r(T-t)}.$$

Since $r > 0$ and $t < T$, this is $< S_t$, the value of Portfolio II at t. So Portfolio I is *always* worth less than Portfolio II if exercised *early.*

If however the option is exercised instead at expiry, T, the American call option is then the same as a European call option. Then (as in Proposition 1 of IV.7): at time T, Portfolio I is worth $\max(S_T, K)$ and Portfolio II is worth S_T – if anything, less; sometimes, more . So:

$$\begin{array}{lll} \text{before} \quad T, & & I < II, \\ \text{at} \quad T, & & I \geq II \text{ always, and} > \text{sometimes.} \end{array}$$

This direct comparison with the underlying [the share in Portfolio II] shows that early exercise is never optimal. Since an American option at expiry is the same as a European one, this completes the proof. // **[10]**
(ii) *Financial Interpretation.*

There are two reasons why an American call should not be exercised early:
1. *Insurance.* Consider an investor choosing to hold a call option instead of the underlying stock. He does not care if the share price falls below the strike price (as he can then just discard his option) – but if he held the stock, he would. Thus the option insures the investor against such a fall in stock price, and if he exercises early, he loses this insurance. **[3]**
2. *Interest on the strike price.* When the holder exercises the option, he buys the stock and pays the strike price, K. Early exercise at $t < T$ loses the interest on K between times t and T: the later he pays out K, the better. **[3]**
(iii) *Economic situation.* Despite this, manufacturers routinely exercise American calls early. If one manufactures (say) tyres, one's raw materials include rubber. The danger is future price increases; the obvious precaution is to hedge against this by buying call options. If stocks run low, exercise early to keep one's production lines running. The insurance aspect is irrelevant: one will *use* the rubber, not sell it. The interest aspect is also irrelevant: manufacturers use their cash to fund their business activity, not to put in the bank, where it would lie comparatively idle. **[4]**
[Seen – lectures]

Q3 (*Wheat options*). The price of wheat now is 134 £/tonne. Next year, it will be 146 or 128, each with positive probability. The strike is $K = 134$.
Risk-neutral measure. We determine p^*, the 'up probability', so as to make the price a martingale. Neglecting interest, this gives

$$134 = p^*.146 + (1 - p^*).128 = 128 + 18p^*, \qquad 6 = 18p^*, \qquad p^* = 1/3.$$

(i) *Pricing.* There is no discounting, so the value V_0 at time 0 is the P^*-expectation E^* of the payoff H next year:

$$V_0 = E^*[H] = p^*.12 + (1 - p^*).0 = 12p^* = 12.1/3 = 4. \qquad \textbf{[6]}$$

(ii) *Hedging.* The call C is financially equivalent to a portfolio Π consisting of a combination of cash and wheat, as the binomial model is *complete* – all contingent claims (options etc.) can be *replicated*. To find *which* combination (ϕ_0, ϕ_1) of cash and wheat, we solve two simultaneous linear equations:

$$\text{Up}: \quad 12 = \phi_0 + 146\phi_1,$$
$$\text{Down}: \quad 0 = \phi_0 + 128\phi_1.$$

Subtract: $12 = 18\phi_1$: $\phi_1 = 2/3$.
Substitute: $\phi_0 = -128\phi_1 = -128 \times 2/3 = -256/3 = -85\ 1/3$.
So C is equivalent to the portfolio $\Pi = (-85\ 1/3, 2/3)$: *long*, 2/3 tonne wheat, *short*, £ 85.33 cash.
Check: in a year's time,
Wheat up: Π is worth (2/3).146 - (2/3).128 = (2/3).18 = 12, as H is;
Wheat down: Π is worth (2/3).128 - (2/3).128 = 0, as H is. **[6]**
Arbitrage. By (i) and (ii), you know C and Π are worth 4 now.
(iii) If you see C being traded (= bought and sold) for *more* than it is worth, *sell* it, for 5. You can buy it, or equivalently the hedging portfolio Π, for 4. Pocket the risk-free profit 1 (pound per option) now. The hedge enables you to meet your obligations to the option holder, at zero net cost. **[2]**
(iv) If C is being traded for *less* than it is worth, *buy* it, for 3. You can sell it, equivalently Π, for 4. Pocket the risk-free profit 1 now. Again, the option payoff enables you to clear your trading account, at zero net cost. **[2]**
(v) Call options of wheat are bought by manufacturers of flour, bread etc., as an insurance policy against prices moving up (e.g., after a bad harvest). **[2]**
(vi) Put options of wheat are bought by growers, as an insurance policy against prices moving down (e.g., after a good harvest). **[2]**
Options of either kind may be traded opportunistically, by speculators.
[Similar seen: lectures and problems]

Q4. *Black-Scholes formula.* (i) There is no point in investing in a risky stock, at mean return rate μ, if one can do as well or better by investing risklessly at rate r. So one should invest all one's funds in cash, unless $\mu > r$. The excess return $\mu - r$ is risky, and σ measures the risk involved. The *Sharpe ratio* is $\lambda := (\mu - r)/\sigma$. This is the usual measure to use, e.g. in deciding between one risky investment and another. **[2]**

By Markowitzian diversification, the manager would wish to have some cash and some stock; he would increase the proportion of his funds held in stock as λ increases. **[1]**

(ii) The Black-Scholes formula gives the value of a European option on a risky stock with dynamics as in $(*)$. One should:

(a) pass from the real-world (or physical) probability measure P to the *risk-neutral* probability measure P^* – the probability measure equivalent to P (same events possible, same events impossible), but under which the dynamics are

$$dS_t = S_t(rdt + \sigma dB_t), \qquad [\mathbf{4}]$$

– i.e., one replaces μ by r;

(b) *discount* (by the riskless interest rate r), so passing from nominal prices to real prices. This replaces the dynamics by

$$dS_t = S_t.\sigma dB_t. \qquad [\mathbf{3}]$$

This can be integrated (stochastic exponential), to give S_T. The Risk-Neutral Valuation Formula gives the option price as the expectation of the payoff (a simple function of S, $(S-K)_+$ or $(K-S)_+$) under P^*. The resulting integral can be evaluated in two terms, both involving Φ, one involving also S_T, the other the discounted strike price K. **[4]**

(iii) The Black-Scholes formula does not involve μ, as it is replaced by r in step (a) above (technically, this is an application of *Girsanov's theorem*). **[3]**

(iv) Here σ is the *volatility* of the stock, a measure of how changeable it is as market conditions change. We do not know it, so have to estimate it. Since the Black-Scholes price is an increasing function of σ ('options like volatility'), one can infer σ (or what the market thinks it is) by matching it to the value giving the price at which the relevant option is currently trading (*implied volatility*). **[2]**

Alternatively, one can look at the price process over time and use Time Series methods from Statistics to estimate σ (*historic volatility*). **[1]**

[Seen – lectures] N. H. Bingham