M3/4/5A22 ASSESSED COURSEWORK SOLUTIONS, 7.12.2015

Quadratic Options: Call. As in L18 (p1, 2nd display – and L28 p4 below), the call price at time $t = 0$ is

$$\begin{aligned} C &= e^{-rT}\int_{-\infty}^{\infty}[S_0\exp\{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}x\}-K]_+^2\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}dx, \\ &= e^{-rT}\int_{c}^{\infty}[S_0\exp\{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}x\}-K]^2\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}dx, \qquad [\mathbf{3}] \end{aligned}$$

where $c := [\log(K/S_0) - (r - \frac{1}{2}\sigma^2)T]/\sigma\sqrt{T}$. Squaring,

$$C = e^{-rT}[C_1 + C_2 + C_3], \qquad [\mathbf{2}]$$

where (recalling $d_\pm$ from L17, 18, and $d_+ - d_- = \sigma\sqrt{T}$)

$$C_1 = S_0^2\int_c^{\infty}\exp\{(2r-\sigma^2)T+2\sigma\sqrt{T}x\}\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}dx = S_0^2e^{2rT}I, \qquad \text{say,} \quad [\mathbf{1}]$$

$$\begin{aligned} C_2 &= -2S_0K\int_c^{\infty}\exp\{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}x-\frac{1}{2}x^2\}dx/\sqrt{2\pi}, \\ &= -2S_0Ke^{rT}\int_c^{\infty}\exp\{-\frac{1}{2}(x-\sigma\sqrt{T})^2\}dx/\sqrt{2\pi} \\ &= -2S_0Ke^{rT}\Phi(d_+) \qquad \text{(as in L18)}, \qquad [\mathbf{3}] \end{aligned}$$

$$C_3 = K^2\int_c^{\infty}\phi(x)dx = K^2\Phi(d_-) = K^2\Phi(d_+ - \sigma\sqrt{T}), \qquad [\mathbf{3}]$$

as $\int_c^\infty \phi(x)dx = 1 - \Phi(c) = \Phi(-c) = \Phi(d_-)$, in the notation of L18. Now

$$\begin{aligned} I &= \int_c^{\infty}\exp\{-\sigma^2T+2\sigma\sqrt{T}x-\frac{1}{2}x^2\}dx/\sqrt{2\pi} \\ &= e^{\sigma^2T}\int_c^{\infty}\exp\{-\frac{1}{2}(x-2\sigma\sqrt{T})^2\}dx/\sqrt{2\pi} = e^{\sigma^2T}\int_{c-2\sigma\sqrt{T}}^{\infty}\phi(u)du \\ &= e^{\sigma^2T}\int_{-\infty}^{2\sigma\sqrt{T}-c}\phi = e^{\sigma^2T}\Phi(2\sigma\sqrt{T}-c) = e^{\sigma^2T}\Phi(d_+ + \sigma\sqrt{T}), \ [\mathbf{6}] \end{aligned}$$

as $-c + 2\sigma\sqrt{T} = d_- + 2\sigma\sqrt{T} = d_+ + \sigma\sqrt{T}$. Combining,

$$C = S_0^2e^{(r+\sigma^2)T}\Phi(d_+ + \sigma\sqrt{T}) - 2S_0K\Phi(d_+) + K^2e^{-rT}\Phi(d_+ - \sigma\sqrt{T}). \quad [\mathbf{2}]$$

NHB