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Lecture 30 19.12.2014

To summarise the basic case (μ and σ constant) in a nutshell:
(i). Dynamics are given by GBM, $dS_t = \mu S dt + \sigma S dW_t$.
(ii). Discount: $d\tilde{S}_t = (\mu - r)\tilde{S} dt + \sigma \tilde{S} dW_t$.
(iii). Use Girsanov's Theorem to change μ to r: under P^*, $d\tilde{S}_t = \sigma \tilde{S} dW_t$.
(iv). Integrate: the RHS gives a P^*-martingale, so has constant E^*-expectation.
Comments.
1. Calculation. When solutions have to be found numerically (as is the case in general - though not for some important special cases such as European call options, considered below), we again have a choice of
(i) analytic methods: numerical solution of a PDE,
(ii) probabilistic methods: evaluation, by the Risk-Neutral Valuation Formula, of an expectation.
A comparison of convenience between these two methods depends on one's experience of numerical computation and the software available. However, in the simplest case considered here, the probabilistic problem involves a one-dimensional integral, while the analytic problem is two-dimensional (involves a two-variable PDE: one variable would give an ODE!). So on dimensional grounds, and because of the probabilistic content of this course, we will generally prefer the probabilistic approach.
2. The Feynman-Kac formula. It is interesting to note that the Feynman-Kac formula originates in an entirely different context, namely quantum physics. In the late 1940s, the physicist Richard Feynman developed his path-integral approach to quantum mechanics, leading to his work (with Schwinger, Tomonaga and Dyson) on QED (quantum electrodynamics). Feynman's approach was non-rigorous; Mark Kac, an analyst and probabilist with an excellent background in PDE, produced a rigorous version which led to the approach above.
3. The Sharpe ratio. There is no point in investing in a risky asset with mean return rate μ, when cash is a riskless asset with return rate r, unless $\mu > r$. The excess return $\mu - r$ is compared with the risk, as measured by the volatility σ via the *Sharpe ratio*

$$\lambda := (\mu - r)/\sigma,$$

also known as the *market price of risk*.
4. The Greeks and delta-hedging. This is much as in discrete time (Ch. IV).
5. Discrete and continuous time. One often has a choice between discrete

and continuous time. For discrete time, we have proved everything; for continuous time, we have had to quote the hard proofs. Note that in continuous time we can use calculus – PDEs, SDEs etc. In discrete time we use instead the calculus of finite differences.
6. The calculus of finite differences. This is very similar to ordinary calculus (old-fashioned name: the *infinitesimal calculus* – thus the opposite of finite here is infinitesimal, not infinite!). It is in some ways harder. For instance: you all know integration by parts (partial integration) backwards. The discrete analogue – partial summation, or Abel's lemma – may be less familiar.

The calculus of finite differences used to be taught for use in e.g. interpolation (how to use information in mathematical tables to 'fill in missing values'). This is now done by computer subroutines – *but*, computers work discretely (with differences rather than derivatives), so the subject is still alive and well.

§5. Further results

1. American Calls.

As in discrete time, these are equivalent to European calls - there is no advantage in early exercise

2. American Puts.

The results on Snell envelopes, least supermartingale majorants etc. extend to continuous time.

Pricing American calls is an *optimal stopping problem*: one wants to choose the exercise time so as to maximise the payoff. There is a whole subject on optimal stopping; see e.g. the book by Peskir & Shiryaev, [PS]. There are links with real (investment) options (below).

3. Exotic options.

The options considered so far (put/call, European/American) are so standard now as to be commonly called *vanilla options.* More complicated types of option are called *exotic options.* These include:
Barrier options, where the payoff depends on whether some barrier has been crossed ('up and in, up and out, down and in, down and out');
Lookback options, where one can retrospectively 'buy at the low, sell at the high'.
The mathematics here is very interesting, but we cannot develop it here.

4. Jumps in stock price.

We mentioned the *jitter* in stock prices: these jump, when looked at closely enough. Also, big trades move prices, and so do economic shocks.

The Black-Scholes model based on BM, which is continuous, cannot handle this. More general processes (Lévy processes – stationary independent increments) are needed here. But these model *incomplete* markets – so prices are no longer unique (one has a bid-ask spread). We stress: *real markets are incomplete*. Real prices jump. The completeness of the Black-Scholes model, and the Brownian Martingale Representation Theorem, reflects the *continuity* of BM.

5. Varying or random interest rates. This is the subject of ongoing research.

6. Transaction costs.

Real markets suffer from friction: there are actual costs in trading and making transactions, which complicate the theory.

7. Higher interest rates for borrowing than lending.

Real financial markets have higher interest rates for borrowing than for lending (which is how banking works), and this introduces another kind of friction into the market. Again: ongoing research.

8. Stochastic volatility (SV).

There are a number of *stylised facts* in mathematical finance. E.g.:

(i). Financial data show *skewness*. This is a result of the asymmetry between profit and loss (large losses are lethal!; large profits are just nice to have).

(ii). Financial data have much *fatter tails* than the normal (Gaussian). We have discussed this in I.5.

(iii) Financial data show *volatility clustering*. This is a result of the economic and financial environment, which is extremely complex, and which moves between good times/booms/upswings and bad times/slumps/downswings. Typically, the market 'gets stuck', staying in its current state for longer than is objectively justified, and then over-correcting.

9. ARCH and GARCH.

These are econometric (time series) models widely used for modelling stochastic volatility and volatility clustering.

10. Real options (Investment options)

These are concerned, not with financial derivatives, but with business decision-making – typically, the decision of whether or not to make a particular investment, and if so, when. Because these options concern the real economy (of manufacturing, etc.) rather than financial markets such as the stock market, such options are often called *real options*. But because they typically concern investment decisions, they are also often called *investment options*. There is a good introductory treatment in [D&P].

Postscript.

1. One recent book on Financial Mathematics describes the subject as being composed of three strands:
arbitrage – the core economic concept, which we have used throughout;
martingales – the key probabilistic concept (Ch. III on);
numerics. Finance houses in the City use *models*, which they need to *calibrate to data* – a task involving both statistical and numerical skills, and in particular an ability to *programme*.

2. You will probably already have experience with at least one general mathematics package (e.g., Mathematica and/or Maple) (if not: get it, a.s.a.p.!). You may also know some Numerical Analysis, the theory behind computation. You may have encountered *simulation*, also known as *Monte Carlo*, and/or a branch of Probability and Statistics called *Markov Chain Monte Carlo (MCMC)* – computer-intensive methods for numerical solutions to problems too complicated to solve analytically. The leaders of R & D teams in the City need to be expert at both stochastic modelling (e.g., to propose new products), and simulation (to evaluate how these perform). Most of the ones I know use Matlab for this. At a lower level, quantitative analysts (quants) working under them need expertise in a computer language; C++ is the industry standard. If you are thinking of a career in Mathematical Finance, learn C++, as soon as possible, and for academic credit.

3. This course deals with *equity markets* – with *stocks*, and financial derivatives of them – options on stocks, etc. The relevant mathematics is *finite-dimensional*. Lurking in the background are *bond markets* ('money markets': bonds, gilts etc., where *interest rates* dominate), and the relevant options – *interest-rate derivatives*, and *foreign exchange* between different currencies ('forex'). The resulting mathematics (which is highly topical, and so in great demand in the City!) is *infinite-dimensional*, and so much harder than the equity-market theory we have done. However, the underlying principles are basically the same. One has to learn to walk before one learns to run, and equity markets serve as a preparation for money markets.

4. The aim of this lecture course is simple. It is to familiarize the student with the basics of Black-Scholes theory, as the core of modern finance, and with the mathematics necessary to understand this. The motivation driving the ever-increasing study of this material is the financial services industry and the City. I hope that any of you who seek City careers will find this introduction to the subject useful in later life. NHB, 2014