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Lecture 29 16.12.2014

This gives the following result:

Theorem (Risk-Neutral Valuation Formula). The no-arbitrage price of the claim $h(S_T)$ is given by

$$F(t,x) = e^{-r(T-t)} E^*_{t,x} h(S_T),$$

where $S_t = x$ is the asset price at time t and P^* is the measure under which the asset price dynamics are given by

$$dS_t = rS_t dt + \sigma(t, S_t) S_t dW_t.$$

Corollary. In the Black-Scholes model above, the arbitrage-free price does not depend on the mean return rate μ of the underlying asset.

Comments.

1. *Risk-neutral measure.* We call P^* the *risk-neutral* probability measure. It is equivalent to P (by Girsanov's Theorem – the change-of-measure result, which deals with change of drift in SDEs – see §4), and is a martingale measure (as the discounted asset prices are P^*-martingales, by above), i.e. P^* (or Q) is the *equivalent martingale measure (EMM).*
2. *Fundamental Theorem of Asset Pricing.* The above continuous-time result may be summarised just as the Fundamental Theorem of Asset Pricing in discrete time: to get the no-arbitrage price of a contingent claim, take the discounted expected value under the equivalent mg (risk-neutral) measure.
3. *Completeness.* In discrete time, we saw that absence of arbitrage corresponded to *existence* of risk-neutral measures, completeness to *uniqueness*. We have obtained existence and uniqueness here (and so completeness), by appealing to existence and uniqueness theorems for PDEs (which we have not proved!). A more probabilistic route is to use Girsanov's Theorem (§4) instead. Completeness questions then become questions on representation theorems for Brownian martingales (§4). As usual, there is a choice of routes to the major results – in this case, a trade-off between analysis (PDEs) and probability (Girsanov's Theorem and the Representation Theorem for Brownian Martingales, §4).

Now the process specified under P^* by the dynamics $(**)$ is our old friend geometric Brownian motion, $GBM(r, \sigma)$. Thus if S_t has P^*-dynamics

$$dS_t = rS_t dt + \sigma S_t dW_t, \qquad S_t = s,$$

with W a P^*-Brownian motion, then we can write S_T explicitly as

$$S_T = s \exp\{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t)\}.$$

Now $W_T - W_t$ is normal $N(0, T-t)$, so $(W_T - W_t)/\sqrt{T-t} =: Z \sim N(0,1)$:

$$S_T = s \exp\{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma Z \sqrt{T-t}\}, \qquad Z \sim N(0,1).$$

So by the Risk-Neutral Valuation Formula, the pricing formula is

$$F(t,x) = e^{-r(T-t)} \int_{-\infty}^{\infty} h(s \exp\{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma(T-t)^{\frac{1}{2}} x\}).\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} dx.$$

For a general payoff function h, there is no explicit formula for the integral, which has to be evaluated numerically. But we can evaluate the integral for the basic case of a European call option with strike-price K:

$$h(s) = (s-K)^+.$$

Then

$$F(t,x) = e^{-r(T-t)} \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} [s \exp\{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma(T-t)^{\frac{1}{2}} x\} - K]_+ dx.$$

We have already evaluated integrals of this type in Chapter IV, where we obtained the Black-Scholes formula from the binomial model by a passage to the limit. Completing the square in the exponential as before gives the

Continuous Black-Scholes Formula.

$$F(t,s) = s\Phi(d_+) - e^{-r(T-t)} K \Phi(d_-),$$

where

$$d_\pm := [\log(s/K) + (r \pm \frac{1}{2}\sigma^2)(T-t)]/\sigma\sqrt{T-t}.$$

§4. Girsanov's Theorem

Consider first ([KS], §3.5) independent $N(0,1)$ random variables $Z_1, \cdots, Z_n$ on $(\Omega, \mathcal{F}, \mathcal{P})$. Given a vector $\mu = (\mu_1, \cdots, \mu_n)$, consider a new probability measure $\tilde{P}$ on $(\Omega, \mathcal{F})$ defined by

$$\tilde{P}(d\omega) = \exp\{\Sigma_1^n \mu_i Z_i(\omega) - \frac{1}{2}\Sigma_1^n \mu_i^2\}.P(d\omega).$$

This is a positive measure as $\exp\{.\} > 0$, and integrates to 1 as $\int \exp\{\mu_i Z_i\} dP = \exp\{\frac{1}{2}\mu_i^2\}$, so is a probability measure. It is also equivalent to P (has the same null sets – actually, the only null set are Lebesgue-null sets, in each case), again as the exponential term is positive. Also

$$
\begin{aligned}
\tilde{P}(Z_i \in dz_i, \quad i = 1, \cdots, n) &= \exp\{\Sigma_1^n \mu_i z_i - \frac{1}{2}\Sigma_1^n \mu_i^2\}.P(Z_i \in dz_i, \quad i = 1, \cdots, n) \\
&= (2\pi)^{-\frac{1}{2}n} \exp\{\Sigma \mu_i z_i - \frac{1}{2}\Sigma \mu_i^2 - \frac{1}{2}\Sigma z_i^2\} \Pi dz_i \\
&= (2\pi)^{-\frac{1}{2}n} \exp\{-\frac{1}{2}\Sigma(z_i - \mu_i)^2\} dz_1 \cdots dz_n.
\end{aligned}
$$

This says that if the Z_i are independent $N(0,1)$ under P, they are independent $N(\mu_i, 1)$ under $\tilde{P}$. Thus the effect of the *change of measure* $P \to \tilde{P}$, from the original measure P to the *equivalent* measure $\tilde{P}$, is to *change the mean*, from $0 = (0, \cdots, 0)$ to $\mu = (\mu_1, \cdots, \mu_n)$.

This result extends to infinitely many dimensions – i.e., from random vectors to stochastic processes, indeed with random rather than deterministic means. We quote (Igor Vladimirovich GIRSANOV (1934-67) in 1960):

Theorem (Girsanov's Theorem). Let $(\mu_t : 0 \le t \le T)$ be an adapted (e.g., left-continuous) process with $\int_0^T \mu_t^2 dt < \infty \quad a.s.$, and such that the process $(L_t : 0 \le t \le T)$ defined by

$$L_t = \exp\{-\int_0^t \mu_s dW_s - \frac{1}{2}\int_0^t \mu_s^2 ds\}$$

is a martingale. Then, under the probability P_L with density L_T relative to P, the process W^* defined by

$$W_t^* := W_t + \int_0^t \mu_s ds, \qquad (0 \le t \le T)$$

is a standard Brownian motion.

Here, L_t is the Radon-Nikodym derivative of P_L w.r.t. P on the σ-algebra $\mathcal{F}_t$. In particular, for $\mu_t \equiv \mu$, change of measure by introducing the RN derivative $\exp\{\mu W_t - \frac{1}{2}\mu^2\}$ corresponds to a change of drift from 0 to μ.

Girsanov's Theorem (or the Cameron-Martin-Girsanov Theorem) is formulated in varying degrees of generality, and proved, in [KS, §3.5], [RY, VIII].

Consider now the Black-Scholes model, with dynamics

$$dB_t = rB_t dt, \qquad dS_t = \mu S_t dt + \sigma S_t dW_t.$$

The discounted asset prices $\tilde{S}_t := e^{-rt} S_t$ have dynamics given, as before, by

$$d\tilde{S}_t = -re^{-rt} S_t dt + e^{-rt} dS_t = -r\tilde{S}_t dt + \mu \tilde{S}_t dt + \sigma \tilde{S}_t dW_t$$
$$= (\mu - r)\tilde{S}_t dt + \sigma \tilde{S}_t dW_t.$$

Now the drift term – the dt term – here prevents $\tilde{S}_t$ being a martingale; the noise – dW_t – term gives a stochastic integral, which is a martingale. Girsanov's theorem suggests the change of measure from P to the equivalent martingale measure (or risk-neutral measure) P^* that makes the discounted asset price a martingale. This
(i) gives directly the continuous-time version of the Fundamental Theorem of Asset Pricing: *to price assets, take expectations of discounted prices under the risk-neutral measure*;
(ii) allows a probabilistic treatment of the Black-Scholes model, avoiding the detour via PDEs of §2, §3.

Theorem (Representation Theorem for Brownian Martingales). Let $(M_t : 0 \leq t \leq T)$ be a square-integrable martingale with respect to the Brownian filtration $(\mathcal{F}_t)$. Then there exists an adapted process $H = (H_t : 0 \leq t \leq T)$ with $E \int H_s^2 ds < \infty$ such that

$$M_t = M_0 + \int_0^t H_s dW_s, \qquad 0 \leq t \leq T.$$

That is, all Brownian martingales may be represented as stochastic integrals with respect to Brownian motion.

We refer to, e.g., [KS], [RY] for proof. The multidimensional version of the result also holds, and may be proved in the same way.

The economic relevance of the Representation Theorem is that it shows that the Black-Scholes model is *complete* – that is, that equivalent martingale measures are unique. Mathematically, the result is purely a consequence of properties of the Brownian filtration. The desirable mathematical properties of Brownian motion are thus seen to have hidden within them desirable economic and financial consequences of real practical value.