m3a22l28tex

Lecture 28 15.12.2014

Proof of the Black-Scholes PDE (continued).

Substituting the values above (L27) in the no-arbitrage relation gives

$$\frac{-SF_2}{F - SF_2}.\mu + \frac{F}{F - SF_2}.\frac{F_1 + \mu SF_2 + \frac{1}{2}\sigma^2 F_{22}}{F} = r.$$

So

$$-SF_2\mu + F_1 + \mu SF_2 + \frac{1}{2}\sigma^2 S^2 F_{22} = rF - rSF_2,$$

giving

$$F_1 + rSF_2 + \frac{1}{2}\sigma^2 S^2 F_{22} - rF = 0. \qquad (BS)$$

This completes the proof of the Black-Scholes PDE. //

Corollary. The no-arbitrage price of the derivative does not depend on the mean return $\mu(t,.)$ of the underlying asset, only on its *volatility* $\sigma(t,.)$ and the short interest-rate.

The Black-Scholes PDE may be solved analytically, or numerically. We give an alternative probabilistic approach below.

The Black-Scholes PDE is parabolic, and can be transformed into the heat equation, whose solution can be written down in terms of an integral and the *heat kernel*. This is the same as the probabilistic solution obtained below.

Note. 1. Black and Scholes were classically trained applied mathematicians. When they derived their PDE, they recognised it as parabolic. After some months' work, they were able to transform it into the heat equation. The solution to this is known classically.[1] On transforming back, they obtained the Black-Scholes formula.

The Black-Scholes formula transformed the financial world. Before it (see

[1] See e.g. the link to MPC2 (Mathematics and Physics for Chemists, Year 2) on my website, Weeks 4, 9. The solution is in terms of *Green functions*. The Green function for (fundamental solution of) the heat equation has the form of a normal density. This reflects the close link between the mathematics of the heat equation (J. Fourier (1768-1830) in 1807; *Théorie analytique de la chaleur* in 1822) and the mathematics of Brownian motion, which as we have seen belongs to the 20th Century. The link was made by S. Kakutani in 1944, and involves potential theory.

Ch. I), the expert view was that asking what an option is worth was (in effect) a silly question: the answer would necessarily depend on the attitude to risk of the individual considering buying the option. It turned out that – at least approximately (i.e., subject to the restrictions to perfect – frictionless – markets, including No Arbitrage – an over-simplification of reality) there *is* an option value. One can see this in one's head, without doing any mathematics, if one knows that the Black-Scholes market is *complete* (see VI.3 below, VI.4 L29). So, every contingent claim (option, etc.) can be *replicated*, in terms of a suitable combination of cash and stock. Anyone can price this:
(i) count the cash, and count the stock;
(ii) look up the current stock price;
(iii) do the arithmetic.
2. The programmable pocket calculator was becoming available around this time. Every trader immediately got one, and programmed it, so that he could price an option (using the Black-Scholes model!) in real time, from market data.
3. The missing quantity in the Black-Scholes formula is the *volatility*, σ. But, the price is continuous and strictly increasing in σ (options like volatility!). So there is *exactly one* value of σ that gives the price at which options are being currently traded. The conclusion is that this is the value that the market currently judges σ to be. This is the value (called the *implied volatility* that traders use.
4. Because the Black-Scholes model is the benchmark model of mathematical finance, and gives a value for σ at the push of a button, it is widely used.
5. This is *despite* the fact that no one actually believes the Black-Scholes model! It gives at best an over-simplified approximation to reality. Indeed, Fischer Black himself famously once wrote a paper called *The holes in Black-Scholes*.
6. This is an interesting example of theory and practice interacting!
7. Black and Scholes has considerable difficulty in getting their paper published! It was ahead of its time. When published, and its importance understood, it changed its times.

§3. The Feynman-Kac Formula, Risk-Neutral Valuation and the Continuous Black-Scholes Formula

Suppose we consider a SDE, with initial condition (IC), of the form

$$dX_s = \mu(s, X_s)ds + \sigma(s, X_s)dW_s \qquad (t \leq s \leq T), \qquad (SDE)$$

$$X_t = x. \tag{IC}$$

For suitably well-behaved functions μ, σ, this SDE has a unique solution $X = (X_s : t \leq s \leq T)$, a diffusion. We refer for details on solutions of SDEs and diffusions to an advanced text such as [RW2], [RY], [KS §5.7]. Uniqueness of solutions of the SDE is related to *completeness*, and uniqueness of prices: see VI.4 L29. This is much as in the FTAP of Ch. IV, but the continuous-time case is harder – we have to quote uniqueness rather than prove it as we did there.

Taking existence of a unique solution for granted for the moment, consider a smooth function $F(s, X_s)$ of it. By Itô's Lemma,

$$dF = F_1 ds + F_2 dX + \frac{1}{2} F_{22} (dX)^2,$$

and as $(dX)^2 = (\mu ds + \sigma dW_s)^2 = \sigma^2 (dW_s)^2 = \sigma^2 ds$, this is

$$dF = F_1 ds + F_2(\mu ds + \sigma dW_s) + \frac{1}{2}\sigma^2 F_{22} ds = (F_1 + \mu F_2 + \frac{1}{2}\sigma^2 F_{22}) ds + \sigma F_2 dW_s. \tag{$*$}$$

Now suppose that F satisfies the PDE, with boundary condition (BC),

$$F_1(t, x) + \mu(t, x) F_2(t, x) + \frac{1}{2}\sigma^2 F_{22}(t, x) = g(t, x) \tag{PDE}$$

$$F(T, x) = h(x). \tag{BC}$$

Then $(*)$ gives

$$dF = g ds + \sigma F_2 dW_s,$$

which can be written in stochastic-integral form as

$$F(T, X_T) = F(t, X_t) + \int_t^T g(s, X_s) ds + \int_t^T \sigma(s, X_s) F_2(s, X_s) dW_s.$$

The stochastic integral on the right is a martingale, so has constant expectation, which must be 0 as it starts at 0. Recalling that $X_t = x$, writing $E_{t,x}$ for expectation with value x and starting-time t, and the price at expiry T as $h(X_T)$ as before, taking $E_{t,x}$ gives

$$E_{t,x} h(X_T) = F(t, x) + E_{t,x} \int_t^T g(s, X_s) ds.$$

This gives:

Theorem (Feynman-Kac Formula). The solution $F = F(t,x)$ to the PDE

$$F_1(t,x) + \mu(t,x)F_2(t,x) + \frac{1}{2}\sigma^2(t,x)F_{22}(t,x) = g(t,x) \qquad (PDE)$$

with final condition $F(T,x) = h(x)$ has the stochastic representation

$$F(t,x) = E_{t,x}h(X_T) - E_{t,x}\int_t^T g(s,X_s)ds, \qquad (FK)$$

where X satisfies the SDE

$$dX_s = \mu(s,X_s)ds + \sigma(s,X_s)dW_s \qquad (t \le s \le T) \qquad (SDE)$$

with initial condition $X_t = x$.

Now replace $\mu(t,x)$ by rx, $\sigma(t,x)$ by σx, g by rF in the Feynman-Kac formula above. The SDE becomes

$$dX_s = rX_s ds + \sigma X_s dW_s \qquad (**)$$

– the same as for a risky asset with mean return-rate r (the short interest-rate for a riskless asset) in place of μ (which disappeared in the Black-Scholes result). The PDE becomes

$$F_1 + rxF_2 + \frac{1}{2}\sigma^2 x^2 F_{22} = rF, \qquad (BS)$$

the Black-Scholes PDE. So by the Feynman-Kac formula,

$$dF = rFds + \sigma F_2 dW_s, \qquad F(T,s) = h(s).$$

We can eliminate the first term on the right by discounting at rate r: write $G(s,X_s) := e^{-rs}F(s,X_s)$ for the discounted price process. Then as before,

$$dG = -re^{-rs}Fds + e^{-rs}dF = e^{-rs}(dF - rFds) = e^{-rs}.\sigma F_2 dW.$$

Then integrating, G is a stochastic integral, so a martingale: *the discounted price process* $G(s,X_s) = e^{-rs}F(s,X_s)$ *is a martingale*, under the measure P^* giving the dynamics in $(**)$. This is the measure P we started with, *except* that μ has been changed to r. Thus, G has constant P^*-expectation:

$$E^*_{t,x}G(t,X_t) = E^*_{t,x}e^{-rt}F(t,X_t) = e^{-rt}F(t,x) = E^*_{T,x}e^{-rT}F(T,X_T) = e^{-rT}h(X_T).$$