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Lecture 25 8.12.2014

Approximation (continued).

It is not possible to include detailed proofs of these assertions in a course of this type [recall that we did not construct the measure-theoretic integral of Ch. II in detail either - and this is harder!]. The key technical ingredient needed is the *Kunita-Watanabe inequalities*. See e.g. [KS], §§3.1-2.

One can define stochastic integration in much greater generality.

1. *Integrands.* The natural class of integrands X to use here is the class of *predictable* processes. These include the left-continuous processes to which we confine ourselves above.

2. *Integrators.* One can construct a closely analogous theory for stochastic integrals with the Brownian integrator B above replaced by a *continuous local martingale* integrator M (or more generally by a *local martingale*: see below). The properties above hold, with D replaced by

$$E[(\int_0^t X_u dM_u)^2] = E\int_0^t X_u^2 d\langle M\rangle_u.$$

See e.g. [KS], [RY] for details.

One can generalise further to *semimartingale* integrators: these are processes expressible as the sum of a local martingale and a process of (locally) finite variation. Now C is replaced by: stochastic integrals of local martingales are local martingales. See e.g. [RW1] or Meyer (1976) for details.

§6. Stochastic Differential Equations (SDEs) and Itô's Lemma

Suppose that U, V are adapted processes,with U locally integrable (so $\int_0^t U_s ds$ is defined as an ordinary integral, as in Ch. II), and V is left-continuous with $\int_0^t EV_u^2 du < \infty$ for all t (so $\int_0^t V_s dB_s$ is defined as a stochastic integral, as in §5). Then

$$X_t := x_0 + \int_0^t U_s ds + \int_0^t V_s dB_s$$

defines a stochastic process X with $X_0 = x_0$. It is customary, and convenient, to express such an equation symbolically in differential form, in terms of the *stochastic differential equation*

$$dX_t = U_t dt + V_t dB_t, \qquad X_0 = x_0. \qquad (SDE)$$

Now suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is a function, continuously differentiable once in its first argument (which will denote time), and twice in its second argument (space): $f \in C^{1,2}$. The question arises of giving a meaning to the stochastic differential $df(t, X_t)$ of the process $f(t, X_t)$, and finding it.

Recall the Taylor expansion of a smooth function of several variables, $f(x_0, x_1, \cdots, x_d)$ say. We use suffices to denote partial derivatives: $f_i := \partial f / \partial x_i$, $\quad f_{i,j} := \partial^2 f / \partial x_i \partial x_j$ (recall that if partials not only exist but are continuous, then the order of partial differentiation can be changed: $f_{i,j} = f_{j,i}$, etc.). Then for $x = (x_0, x_1, \cdots, x_d)$ near u,

$$f(x) = f(u) + \Sigma_{i=0}^{d} (x_i - u_i) f_i(u) + \frac{1}{2} \Sigma_{i,j=0}^{d} (x_i - u_i)(x_j - u_j) f_{i,j}(u) + \cdots$$

In our case (writing t_0 in place of 0 for the starting time):

$$f(t, X_t) = f(t_0, X(t_0)) + (t - t_0) f_1(t_0, X(t_0)) + (X(t) - X(t_0)) f_2 + \frac{1}{2}(t - t_0)^2 f_{11} +$$

$$(t - t_0)(X(t) - X(t_0)) f_{12} + \frac{1}{2}(X(t) - X(t_0))^2 f_{22} + \cdots,$$

which may be written symbolically as

$$df(t, X(t)) = f_1 dt + f_2 dX + \frac{1}{2} f_{11} (dt)^2 + f_{12} dt dX + \frac{1}{2} f_{22} (dX)^2 + \cdots.$$

In this, we
(i) substitute $dX_t = U_t dt + V_t dB_t$ from above,
(ii) substitute $(dB_t)^2 = dt$, i.e. $|dB_t| = \sqrt{dt}$, from §4:

$$df = f_1 dt + f_2 (U dt + V dB) + \frac{1}{2} f_{11} (dt)^2 + f_{12} dt (U dt + V dB) + \frac{1}{2} f_{22} (U dt + V dB)^2 + \cdots$$

Now using $(dB)^2 = dt$,

$$\begin{aligned} (U dt + V dB)^2 &= V^2 dt + 2UV dt dB + U^2 (dt)^2 \\ &= V^2 dt + \text{higher-order terms} : \end{aligned}$$

$$df = (f_1 + U f_2 + \frac{1}{2} V^2 f_{22}) dt + V f_2 dB + \text{higher-order terms}.$$

Summarising, we obtain *Itô's Lemma*, the analogue for the Itô or stochastic calculus of the chain rule for ordinary (Newton-Leibniz) calculus:

Theorem (Itô's Lemma). If X_t has stochastic differential

$$dX_t = U_t dt + V_t dB_t, \qquad X_0 = x_0,$$

and $f \in C^{1,2}$, then $f = f(t, X_t)$ has stochastic differential

$$df = (f_1 + Uf_2 + \frac{1}{2}V^2 f_{22})dt + Vf_2 dB_t.$$

That is, writing f_0 for $f(0, x_0)$, the initial value of f,

$$f(t, X_t)) = f_0 + \int_0^t (f_1 + Uf_2 + \frac{1}{2}V^2 f_{22})dt + \int_0^t Vf_2 dB.$$

This important result may be summarised as follows: use Taylor's theorem formally, together with the rule

$$(dt)^2 = 0, \qquad dtdB = 0, \qquad (dB)^2 = dt.$$

Itô's Lemma extends to higher dimensions, as does the rule above:

$$df = (f_0 + \Sigma_{i=1}^d U_i f_i + \frac{1}{2}\Sigma_1^d V_i^2 f_{ii})dt + \Sigma_1^d V_i f_i dB_i$$

(where U_i, V_i, B_i denote the ith coordinates of vectors U, V, B, f_i, f_{ii} denote partials as above); here the formal rule is

$$(dt)^2 = 0, \qquad dtdB_i = 0, \qquad (dB_i)^2 = dt, \qquad dB_i dB_j = 0 \qquad (i \neq j).$$

Corollary. $Ef(t, X_t) = f_0 + \int_0^t E[f_1 + Uf_2 + \frac{1}{2}V^2 f_{22}]dt$.

Proof. $\int_0^t Vf_2 dB$ is a stochastic integral, so a martingale, so its expectation is constant (= 0, as it starts at 0). //

Note. Powerful as it is in the setting above, Itô's Lemma really comes into its own in the more general setting of semimartingales. It says there that if X is a semimartingale and f is a smooth function as above, then $f(t, X(t))$ is also a semimartingale. The ordinary differential dt gives rise to the bounded-variation part, the stochastic differential gives rise to the martingale part. This closure property under very general non-linear operations is very powerful and important.

Example: The Ornstein-Uhlenbeck Process.

The most important example of a SDE for us is that for geometric Brownian motion (VI.1 below). We close here with another example.

Consider now a model of the velocity V_t of a particle at time t ($V_0 = v_0$), moving through a fluid or gas, which exerts
(i) a frictional drag, assumed propertional to the velocity,
(ii) a noise term resulting from the random bombardment of the particle by the molecules of the surrounding fluid or gas. The basic model is the SDE

$$dV = -\beta V dt + c dB, \qquad (OU)$$

whose solution is called the *Ornstein-Uhlenbeck* (velocity) process with *relaxation time* $1/\beta$ and *diffusion coefficient* $D := \frac{1}{2}c^2/\beta^2$. It is a stationary Gaussian Markov process (not stationary-increments Gaussian Markov like Brownian motion), whose limiting (ergodic) distribution is $N(0, \beta D)$ and whose limiting correlation function is $e^{-\beta|\cdot|}$.

If we integrate the OU velocity process to get the OU *displacement process*, we lose the Markov property (though the process is still Gaussian). Being non-Markov, the resulting process is much more difficult to analyse.

The OU process is the prototype of processes exhibiting *mean reversion*, or a *central push*: frictional drag acts as a restoring force tending to push the process back towards its mean. It is important in many areas, including
(i) statistical mechanics, where it originated,
(ii) mathematical finance, where it appears in the *Vasicek model* for the term-structure of interest-rates (the mean represents the 'natural' interest rate),
(iii) *stochastic volatility* models, where the volatility σ itself is now a stochastic process σ_t, subject to an SDE of OU type.

Theory of interest rates.

This subject dominates the mathematics of *money market*, or *bond markets*. These are more important in today's world than stock markets, but are more complicated, so we must be brief here. The area is crucially important in *macro-economic policy*, and in political decision-making, particularly after the financial crisis ("credit crunch"). Government policy is driven by fear of speculators in the bond markets (rather than aimed at inter-governmental cooperation against them). The mathematics is infinite-dimensional (at each time-point t we have a whole *yield curve* over future times), but reduces to finite-dimensionality: bonds are only offered at discrete times, with a *tenor* structure (a finite set of maturity times).