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§5. Stochastic Integrals (Itô Calculus)

Stochastic integration was introduced by K. ITÔ in 1944, hence its name Itô calculus. It gives a meaning to $\int_0^t XdY = \int_0^t X_s(\omega)dY_s(\omega)$, for suitable stochastic processes X and Y, the *integrand* and the *integrator*. We shall confine our attention here to the basic case with integrator Brownian motion: $Y = B$. Much greater generality is possible: for Y a continuous martingale, see [KS] or [RY]; for a systematic general treatment, see
MEYER, P.-A. (1976): Un cours sur les intégrales stochastiques. *Séminaire de Probabilités X: Lecture Notes on Math.* **511**, 245-400, Springer.

The first thing to note is that stochastic integrals with respect to Brownian motion, *if* they exist, must be *quite different* from the measure-theoretic integral of Ch. II.2. For, the Lebesgue-Stieltjes integrals described there have as integrators the difference of two monotone (increasing) functions (by Jordan's theorem), which are locally of *finite (bounded) variation, FV*. But we know from §4 that Brownian motion is of *infinite (unbounded)* variation on every interval. So Lebesgue-Stieltjes and Itô integrals must be fundamentally different.

In view of the above, it is quite surprising that Itô integrals can be defined at all. But if we take for granted Itô's fundamental insight that they *can* be, it is obvious how to begin and clear enough how to proceed. We begin with the simplest possible integrands X, and extend successively much as we extended the measure-theoretic integral of Ch. II.

1. *Indicators.*
If $X_t(\omega) = I_{[a,b]}(t)$, there is exactly one plausible way to define $\int XdB$:

$$\int_0^t XdB, \quad \text{or} \quad \int_0^t X_s(\omega)dB_s(\omega), := \begin{cases} 0 & \text{if } t \leq a, \\ B_t - B_a & \text{if } a \leq t \leq b, \\ B_b - B_a & \text{if } t \geq b. \end{cases}$$

2. *Simple functions.* Extend by linearity: if X is a linear combination of indicators, $X = \Sigma c_i I_{[a_i,b_i]}$, we should define

$$\int_0^t XdB := \Sigma c_i \int_0^t I_{[a_i,b_i]}dB.$$

Already one wonders how to extend this from constants c_i to suitable random variables, and one seeks to simplify the obvious but clumsy three-line expressions above. It turns out that finite sums are not essential: one can have infinite sums, but now we take the c_i uniformly bounded.

We begin again, this time calling a *stochastic process* X *simple* if there is an infinite sequence

$$0 = t_0 < t_1 < \cdots < t_n < \cdots \to \infty$$

and uniformly bounded $\mathcal{F}_{t_n}$-measurable random variables ξ_n ($|\xi_n| \leq C$ for all n and ω, for some C) if $X_t(\omega)$ can be written in the form

$$X_t(\omega) = \xi_0(\omega) I_{\{0\}}(t) + \Sigma_{i=0}^{\infty} \xi_i(\omega) I_{(t_i, t_{i+1}]}(t) \qquad (0 \leq t < \infty, \omega \in \Omega).$$

The only definition of $\int_0^t X dB$ that agrees with the above for finite sums is, if n is the unique integer with $t_n \leq t < t_{n+1}$,

$$\begin{aligned} I_t(X) := \int_0^t X dB &= \Sigma_0^{n-1} \xi_i (B(t_{i+1}) - B(t_i)) + \xi_n (B(t) - B(t_n)) \\ &= \Sigma_0^{\infty} \xi_i (B(t \wedge t_{i+1}) - B(t \wedge t_i)) \qquad (0 \leq t < \infty). \end{aligned}$$

We note here some properties of the stochastic integral defined so far:

A. $I_0(X) = 0 \qquad P - a.s.$

B. *Linearity.* $I_t(aX + bY) = aI_t(X) + bI_t(Y)$.
Proof. Linear combinations of simple functions are simple.

C. $E[I_t(X)|\mathcal{F}_s] = I_s(X) \qquad P - a.s. \qquad (0 \leq s < t < \infty):$
$I_t(X)$ is a *continuous martingale.*
Proof. There are two cases to consider.
(i) Both s and t belong to the same interval $[t_n, t_{n+1})$. Then

$$I_t(X) = I_s(X) + \xi_n (B(t) - B(s)).$$

But ξ_n is $\mathcal{F}_{t_n}$-measurable, so $\mathcal{F}_s$-measurable ($t_n \leq s$), so independent of $B(t) - B(s)$ (independent increments property of B). So

$$E[I_t(X)|\mathcal{F}_s] = I_s(X) + \xi_n E[B(t) - B(s)|\mathcal{F}_s] = I_s(X).$$

(ii) $s < t$ and t belong to different intervals: $s \in [t_m, t_{m+1})$ for $m < n$. Then

$$\begin{aligned} E[I_t(x)|\mathcal{F}_s] &= E(E[I_t(X)|\mathcal{F}_{t_n}]|\mathcal{F}_s) \qquad \text{(iterated conditional expectations)} \\ &= E(I_{t_n}(X)|\mathcal{F}_s), \end{aligned}$$

since ξ_n $\mathcal{F}_{t_n}$-measurable and independent increments of B give

$$E[\xi_n(B(t) - B(t_n))|\mathcal{F}_{t_n}] = \xi_n E[B(t) - B(t_n)|\mathcal{F}_{t_n}] = \xi_n.0 = 0.$$

Continuing in this way, we can reduce successively to t_{m+1}:

$$E[I_t(X)|\mathcal{F}_s] = E[I_{t_m}(X)|\mathcal{F}_s].$$

But $I_{t_m}(X) = I_s(X) + \xi_m(B(s) - B(t_m))$; taking $E[.|\mathcal{F}_s]$ the second term gives zero as above, giving the result. //

Note. The stochastic integral for simple integrands is essentially a martingale transform, and the above is essentially the proof of Ch. III that martingale transforms are martingales.

We pause to note a property of martingales which we shall need below. Call $X_t - X_s$ the *increment* of X over $(s, t]$. Then for a *martingale* X, *the product of the increments over disjoint intervals has zero mean.* For, if $s < t \le u < v$,

$$\begin{aligned} E[(X_v - X_u)(X_t - X_s)] &= E[E[(X_v - X_u)(X_t - X_s)|\mathcal{F}_u]] \\ &= E[(X_t - X_s)E[(X_v - X_u)|\mathcal{F}_u]], \end{aligned}$$

taking out what is known (as $s, t \le u$). The inner expectation is zero by the martingale property, so the LHS is zero, as required.

D (*Itô isometry*). $E[(I_t(X))^2]$, or $E[(\int_0^t X_s dB_s)^2]$, $= E\int_0^t X_s^2 ds$.
Proof. The LHS above is $E[I_t(X).I_t(X)]$, i.e.

$$E[(\Sigma_{i=0}^{n-1}\xi_i(B(t_{i+1}) - B(t_i)) + \xi_n(B(t) - B(t_n)))^2].$$

Expanding the square, the cross-terms have expectation zero by above, so

$$E[\Sigma_{i=0}^{n-1}\xi_i^2(B(t_{i+i} - B(t_i))^2 + \xi_n^2(B(t) - B(t_n))^2].$$

Since ξ_i is $\mathcal{F}_{t_i}$-measurable, each ξ_i^2-term is independent of the squared Brownian increment term following it, which has expectation $var(B(t_{i+1})-B(t_i)) = t_{i+1} - t_i$. So we obtain

$$\Sigma_{i=0}^{n-1} E[\xi_i^2](t_{i+1} - t_i) + E[\xi_n^2](t - t_n).$$

This is $\int_0^t E[X_u^2]du = E\int_0^t X_u^2 du$, as required.

E. *Itô isometry (continued).* $I_t(X) - I_s(X) = \int_s^t X_u dB_u$ satisfies

$$E[(\int_s^t X_u dB_u)^2] = E[\int_s^t X_u^2 du] \qquad P-a.s.$$

Proof: as above.

F. *Quadratic variation.* The QV of $I_t(X) = \int_0^t X_u dB_u$ is $\int_0^t X_u^2 du$.

This is proved in the same way as the case $X \equiv 1$, that B has quadratic variation process t.

Integrands.

The properties above suggest that $\int_0^t X dB$ should be defined only for processes with

$$\int_0^t E X_u^2 du < \infty \qquad \text{for all} \quad t.$$

We shall restrict attention to such X in what follows. This gives us an L_2-theory of stochastic integration (compare the L_2-spaces introduced in Ch. II), for which Hilbert-space methods are available.

3. *Approximation.*

Recall steps 1 (indicators) and 2 (simple integrands). By analogy with the integral of Ch. II, we seek a suitable class of integrands suitably approximable by simple integrands. It turns out that:

(i) The suitable class of integrands is the class of left-continuous adapted processes X with $\int_0^t E X_u^2 du < \infty$ for all $t > 0$ (or all $t \in [0, T]$ with finite time-horizon T, as here),

(ii) Each such X may be approximated by a sequence of simple integrands X_n so that the stochastic integral $I_t(X) = \int_0^t X dB$ may be defined as the limit of $I_t(X_n) = \int_0^t X_n dB$,

(iii) The stochastic integral $\int_0^t X dB$ so defined still has properties A-F above.