m3a22l23tex
Lecture 23 2.12.2014

Corollary. X is *self-similar* (reproduces itself under scaling), so a Brownian path $X(.)$ is a *fractal*. So too is the zero-set Z.

Brownian motion owes part of its importance to belonging to *all* the important classes of stochastic processes: it is (strong) Markov, a (continuous) martingale, Gaussian, a diffusion, a Lévy process (process with stationary independent increments), etc.

§4. Quadratic Variation (QV) of Brownian Motion; Itô's Lemma

Recall that for ξ $N(\mu, \sigma^2)$, ξ has moment-generating function (MGF)

$$M(t) := E\exp\{t\xi\} = \exp\{\mu t + \frac{1}{2}\sigma^2 t^2\}.$$

Take $\mu = 0$ below; for ξ $N(0, \sigma^2)$,

$$\begin{aligned} M(t) := E\exp\{t\xi\} &= \exp\{\frac{1}{2}\sigma^2 t^2\} \\ &= 1 + \frac{1}{2}\sigma^2 t^2 + \frac{1}{2!}(\frac{1}{2}\sigma^2 t^2)^2 + O(t^6) \\ &= 1 + \frac{1}{2!}\sigma^2 t^2 + \frac{3}{4!}\sigma^4 t^4 + O(t^6). \end{aligned}$$

So as the Taylor coefficients of the MGF are the moments (hence the name MGF!),

$$E(\xi^2) = var\xi = \sigma^2, \qquad E(\xi^4) = 3\sigma^4, \qquad \text{so} \qquad var(\xi^2) = E(\xi^4) - [E(\xi^2)]^2 = 2\sigma^4.$$

For B BM, this gives in particular

$$EB_t = 0, \qquad varB_t = t, \qquad E[(B_t)^2] = t, \qquad var[(B_t)^2] = 2t^2.$$

In particular, for $t > 0$ *small*, this shows that the variance of B_t^2 is negligible compared with its expected value. Thus, the *randomness* in B_t^2 is negligible compared to its mean for t small.

This suggests that if we take a fine enough partition $\mathcal{P}$ of $[0, T]$ – a finite set of points

$$0 = t_0 < t_1 < \cdots < t_k = T$$

with $|\mathcal{P}| := \max |t_i - t_{i-1}|$ small enough – then writing

$$\Delta B(t_i) := B(t_i) - B(t_{i-1}), \qquad \Delta t_i := t_i - t_{i-1},$$

$\Sigma(\Delta B(t_i))^2$ will closely resemble $\Sigma E[(\Delta B(t_i)^2]$, which is $\Sigma \Delta t_i = \Sigma(t_i - t_{i-1}) = T$. This is in fact true a.s.:

$$\Sigma(\Delta B(t_i))^2 \to \Sigma \Delta t_i = T \qquad \text{as} \qquad \max |t_i - t_{i-1}| \to 0.$$

This limit is called the *quadratic variation* V_T^2 of B over $[0, T]$:

Theorem. The quadratic variation of a Brownian path over $[0, T]$ exists and equals T, a.s.

For details of the proof, see e.g. [BK], §5.3.2, SP L22, SA L7,8.

If we increase t by a small amount to $t + dt$, the increase in the QV can be written symbolically as $(dB_t)^2$, and the increase in t is dt. So, formally we may summarise the theorem as

$$(dB_t)^2 = dt.$$

Suppose now we look at the *ordinary* variation $\Sigma|\Delta B_t|$, rather than the *quadratic* variation $\Sigma(\Delta B_t)^2$. Then instead of $\Sigma(\Delta B_t)^2 \sim \Sigma \Delta t \sim t$, we get $\Sigma|\Delta B_t| \sim \Sigma\sqrt{\Delta t}$. Now for Δt small, $\sqrt{\Delta t}$ is of a larger order of magnitude that Δt. So if $\Sigma \Delta t = t$ converges, $\Sigma\sqrt{\Delta t}$ diverges to $+\infty$. This suggests – what is in fact true – the

Corollary. The paths of Brownian motion are of infinite variation - their variation is $+\infty$ on every interval, a.s.

The QV result above leads to Lévy's 1948 result, the Martingale Characterization of BM. Recall that B_t is a continuous martingale with respect to its natural filtration $(\mathcal{F}_t)$ and with QV t. There is a remarkable converse; we give two forms.

Theorem (Lévy; Martingale Characterization of Brownian Motion). If M is any continuous local $(\mathcal{F}_t)$-martingale with $M_0 = 0$ and quadratic variation t, then M is an $(\mathcal{F}_t)$-Brownian motion.

Theorem (Lévy). If M is any continuous $(\mathcal{F}_t)$-martingale with $M_0 = 0$ and $M_t^2 - t$ a martingale, then M is an $(\mathcal{F}_t)$-Brownian motion.

For proof, see e.g. [RW1], I.2. Observe that for $s < t$,

$$B_t^2 = [B_s + (B_t - B_s)]^2 = B_s^2 + 2B_s(B_t - B_s) + (B_t - B_s)^2,$$

$$E[B_t^2|\mathcal{F}_s] = B_s^2 + 2B_s E[(B_t - B_s)|\mathcal{F}_s] + E[(B_t - B_s)^2|\mathcal{F}_s] = B_s^2 + 0 + (t-s):$$

$$E[B_t^2 - t|\mathcal{F}_s] = B_s^2 - s:$$

$B_t^2 - t$ is a martingale.

Quadratic Variation (QV).

The theory above extends to *continuous* martingales (bounded continuous martingales in general, but we work on a finite time-interval $[0, T]$, so continuity implies boundedness). We quote (for proof, see e.g. [RY], IV.1):

Theorem. A continuous martingale M is of finite quadratic variation $\langle M \rangle$, and $\langle M \rangle$ is the unique continuous increasing adapted process vanishing at zero with $M^2 - \langle M \rangle$ a martingale.

Corollary. A continuous martingale M has infinite variation.

Quadratic Covariation. We write $\langle M, M \rangle$ for $\langle M \rangle$, and extend $\langle\ \rangle$ to a bilinear form $\langle ., . \rangle$ with two different arguments by the *polarization identity*:

$$\langle M, N \rangle := \frac{1}{4}(\langle M+N, M+N \rangle - \langle M-N, M-N \rangle).$$

If N is of *finite* variation, $M \pm N$ has the same QV as M, so $\langle M, N \rangle = 0$.

Itô's Lemma. We discuss Itô's Lemma in more detail in §6 below; we pause here to give the link with quadratic variation and covariation. We quote: if $f(t, x_1, \cdots, x_d)$ is C^1 in its zeroth (time) argument t and C^2 in its remaining d space arguments x_i, and $M = (M^1, \cdots, M^d)$ is a continuous vector martingale, then (writing f_i, f_{ij} for the first partial derivatives of f with respect to its ith argument and the second partial derivatives with respect to the ith and jth arguments) $f(M_t)$ has stochastic differential

$$df(M_t) = f_0(M)dt + \Sigma_{i=1}^d f_i(M_t)dM_t^i + \frac{1}{2}\Sigma_{i,j=1}^d f_{ij}(M_t)d\langle M^i, M^j\rangle_t.$$

Integration by Parts. If $f(t, x_1, x_2) = x_1 x_2$, we obtain

$$d(MN)_t = NdM_t + MdN_t + \frac{1}{2}\langle M, N\rangle_t.$$

Similarly for stochastic integrals (defined below): if $Z_i := \int H_i dM_i$ $(i = 1, 2)$, $d\langle Z_1, Z_2\rangle = H_1 H_2 d\langle M_1, M_2\rangle$.

Note. The integration-by-parts formula – a special case of Itô's Lemma, as above – is in fact *equivalent* to Itô's Lemma: either can be used to derive the other. Rogers & Williams [RW1, IV.32.4] describe the integration-by-parts formula/Itô's Lemma as 'the cornerstone of stochastic calculus'.

Fractals Everywhere.

As we saw, a Brownian path is a *fractal* – a *self-similar* object. So too is its zero-set Z. Fractals were studied, named and popularised by the French mathematician Benôit B. Mandelbrot (1924-2010). See his books, and Michael F. Barnsley: *Fractals everywhere*. Academic Press, 1988.

Fractals *look the same at all scales* – diametrically opposite to the familiar functions of Calculus. In Differential Calculus, a differentiable function has a tangent; this means that locally, its graph *looks straight*; similarly in Integral Calculus. While most continuous functions we encounter are differentiable, at least piecewise (i.e., except for 'kinks'), there is a sense in which the typical, or generic, continuous function is *nowhere differentiable*. Thus Brownian paths may look pathological at first sight – but in fact they are typical!

Hedging in continuous time.

Imagine hedging an option in continuous time. In discrete time, this involves repeatedly rebalancing our portfolio between cash and stock; in continuous time, this has to be done continuously. The relevant stochastic processes (Ch. VI) are *geometric Brownian motion (GBM)*, relatives of BM, which, like BM, have *infinite variation* (finite QV). This makes the rebalancing problematic – indeed, impossible in these terms. Analogy: a cyclist has to rebalance continuously, but does so smoothly, not with infinite variation! Or, think of continuous-time control of a manned space-craft (Kalman filter). In practice, hedging has to be done discretely (as in Ch. IV). Or, we can use price processes with *jumps* – finite variation, but now the markets are incomplete.