m3a22l22tex
Lecture 22 1.12.2014
4. Diffusions.

A diffusion is a path-continuous strong-Markov process such that for each time t and state x the following limits exist:

$$\mu(t,x) := \lim_{h\downarrow 0}\frac{1}{h}E[(X_{t+h}-X_t)|X_t=x],$$

$$\sigma^2(t,x) := \lim_{h\downarrow 0}\frac{1}{h}E[(X_{t+h}-X_t)^2|X_t=x].$$

Then $\mu(t,x)$ is called the *drift*, $\sigma^2(t,x)$ the *diffusion coefficient*.

§3. Brownian Motion.

The Scottish botanist Robert Brown observed pollen particles in suspension under a microscope in 1828 and 1829 (though others had observed the phenomenon before him),[1] and observed that they were in constant irregular motion.

In 1900 L. Bachelier considered Brownian motion a possible model for stock-market prices:
BACHELIER, L. (1900): Théorie de la spéculation. *Ann. Sci. Ecole Normale Supérieure* **17**, 21-86
– the first time Brownian motion had been used to model financial or economic phenomena, and before a mathematical theory had been developed.

In 1905 Albert Einstein considered Brownian motion as a model of particles in suspension, and used it to estimate *Avogadro's number* ($N \sim 6 \times 10^{23}$), based on the diffusion coefficient D in the *Einstein relation*

$$var X_t = Dt \qquad (t > 0).$$

In 1923 Norbert WIENER defined and constructed Brownian motion rigorously for the first time. The resulting stochastic process is often called the *Wiener process* in his honour, and its probability measure (on path-space) is called *Wiener measure.*

We define *standard Brownian motion* on $\mathbb{R}$, BM or $BM(\mathbb{R})$, to be a stochastic process $X = (X_t)_{t\geq 0}$ such that

[1] The Roman author Lucretius observed this phenomenon in the gaseous phase – dust particles dancing in sunbeams – in antiquity: *De rerum naturae*, c. 50 BC.

1. $X_0 = 0$,
2. X has *independent increments*: $X_{t+u} - X_t$ is independent of $\sigma(X_s : s \leq t)$ for $u \geq 0$,
3. X has *stationary increments*: the law of $X_{t+u} - X_t$ depends only on u,
4. X has *Gaussian increments*: $X_{t+u} - X_t$ is normally distributed with mean 0 and variance u,

$$X_{t+u} - X_t \sim N(0, u),$$

5. X has *continuous paths*: X_t is a continuous function of t, i.e. $t \mapsto X_t$ is continuous in t.

For time t in a finite interval – $[0, 1]$, say – we can use the following filtered space:
$\Omega = C[0, 1]$, the space of all continuous functions on $[0, 1]$.
The points $\omega \in \Omega$ are thus random functions, and we use the coordinate mappings: X_t, or $X_t(\omega)$, $= \omega_t$.
The filtration is given by $\mathcal{F}_t := \sigma(X_s : 0 \leq s \leq t)$, $\mathcal{F} := \mathcal{F}_1$.
P is the measure on $(\Omega, \mathcal{F})$ with finite-dimensional distributions specified by the restriction that the increments $X_{t+u} - X_t$ are stationary independent Gaussian $N(0, u)$.

Theorem (WIENER, 1923). Brownian motion exists.

The best way to prove this is by construction, and one that reveals some properties. The proof that follows is originally due to Paley, Wiener and Zygmund (1933) and Lévy (1948), but is re-written in the modern language of *wavelet* expansions. We omit details; for these, see e.g. [BK] 5.3.1, or SP l20-22. The Haar system $(H_n) = (H_n(.))$ is a complete orthonormal system (cons) of functions in $L^2[0, 1]$. The Schauder System $\Delta_n)$ is obtained by integrating the Haar system. Consider the triangular function (or 'tent function')

$$\Delta(t) = \begin{cases} 2t & \text{on } [0, \frac{1}{2}), \\ 2(1-t) & \text{on } [\frac{1}{2}, 1], \\ 0 & \text{else.} \end{cases}$$

Write $\Delta_0(t) := t$, $\Delta_1(t) := \Delta(t)$, and define the nth *Schauder function* Δ_n by

$$\Delta_n(t) := \Delta(2^j t - k) \qquad (n = 2^j + k \geq 1).$$

Note that Δ_n has support $[k/2^j, (k+1)/2^j]$ (so is 'localized' on this dyadic interval, which is small for n, j large). We see that

$$\int_0^t H(u)du = \frac{1}{2}\Delta(t),$$

and similarly

$$\int_0^t H_n(u)du = \lambda_n \Delta_n(t),$$

where $\lambda_0 = 1$ and for $n \geq 1$,

$$\lambda_n = \frac{1}{2} \times 2^{-j/2} \qquad (n = 2^j + k \geq 1).$$

The Schauder system (Δ_n) is again a complete orthogonal system on $L^2[0,1]$. We can now formulate the next result; for proof, see the references above.

Theorem (PWZ theorem: Paley-Wiener-Zygmund, 1933). For $(Z_n)_0^\infty$ independent $N(0,1)$ random variables, λ_n, Δ_n as above,

$$W_t := \sum_{n=0}^{\infty} \lambda_n Z_n \Delta_n(t)$$

converges uniformly on $[0,1]$, a.s. The process $W = (W_t : t \in [0,1])$ is Brownian motion.

Thus the above description does indeed define a stochastic process $X = (X_t)_{t \in [0,1]}$ on $(C[0,1], \mathcal{F}, (\mathcal{F}_t), P)$. The construction gives X on $C[0,n]$ for each $n = 1, 2, \cdots$, and combining these: X exists on $C[0,\infty)$. It is also unique (a stochastic process is uniquely determined by its finite-dimensional distributions and the restriction to path-continuity).

No construction of Brownian motion is easy: one needs both some work and some knowledge of measure theory. However, *existence* is really all we need, and this we shall take for granted. For background, see any measure-theoretic text on stochastic processes. The classic is Doob's book, quoted above (see VIII.2 there). Excellent modern texts include Karatzas & Shreve [KS] (see particularly §2.2-4 for construction and §5.8 for applications to economics), Revuz & Yor [RY], Rogers & Williams [RW1] (Ch. 1), [RW2] Itô calculus – below).

We shall henceforth denote standard Brownian motion $BM(\mathbb{R})$ – or just BM for short – by $B = (B_t)$ (B for Brown), though $W = (W_t)$ (W for Wiener) is also common. Standard Brownian motion $BM(\mathbb{R}^d)$ in d dimensions is defined by $B(t) := (B_1(t), \cdots, B_d(t))$, where $B_1, \cdots, B_d$ are *independent* standard Brownian motions in one dimension (*independent copies* of $BM(\mathbb{R})$).

Zeros.

It can be shown that Brownian motion *oscillates*:

$$\limsup_{t\to\infty} X_t = +\infty, \qquad \liminf_{t\to\infty} X_t = -\infty \qquad a.s.$$

Hence, for every n there are zeros (times t with $X_t = 0$) of X with $t \geq n$ (indeed, infinitely many such zeros). So if

$$Z := \{t \geq 0 : X_t = 0\}$$

denotes the zero-set of $BM(\mathbb{R})$:
1. Z is an *infinite* set.
Next, if t_n are zeros and $t_n \to t$, then by path-continuity $B(t_n) \to B(t)$; but $B(t_n) = 0$, so $B(t) = 0$:
2. Z is a *closed* set (Z contains its limit points).
Less obvious are the next two properties:
3. Z is a *perfect* set: every point $t \in Z$ is a limit point of points in Z. So there are *infinitely many* zeros in *every* neighbourhood of *every* zero (so the paths must oscillate amazingly fast!).
4. Z is a (Lebesgue) *null* set: Z has Lebesgue measure zero.

In particular, the diagram above (or any other diagram!) grossly distorts Z: *it is impossible to draw a realistic picture of a Brownian path.*

Brownian Scaling.

For each $c \in (0, \infty)$, $X(c^2 t)$ is $N(0, c^2 t)$, so $X_c(t) := c^{-1} X(c^2 t)$ is $N(0, t)$. Thus X_c has all the defining properties of a Brownian motion (check). So, X_c **IS** a Brownian motion:

Theorem. If X is BM and $c > 0$, $X_c(t) := c^{-1} X(c^2 t)$, then X_c is again a BM.