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Lecture 20 25.10.2014

Put-Call Symmetry.

The BS formulae for puts and calls resemble each other, with stock price S and discounted strike K interchanged. Results of this type are called *put-call symmetry.*

American Puts.

Recall the put-call parity of Ch. I (valid only for European options):

$$c - p = S - Ke^{-rT}.$$

A partial analogue for American options is given by the inequalities below:

$$S - K < C - P < S - Ke^{-rT}.$$

For proof (as above) and background, see e.g. Ch. 8 (p. 216) of [H1].

We now consider how to evaluate an American put option, European and American call options having been treated already. First, we will need to work in discrete time. We do this by dividing the time-interval $[0, T]$ into N equal subintervals of length Δt say. Next, we take the values of the underlying stock to be discrete: we use the binomial model of §5, with a slight change of notation: we write u, d ('up', 'down') for $(1 + b), (1 + a)$: thus stock with initial value S is worth $Su^i d^j$ after i steps up and j steps down. Consequently, after N steps, there are $N + 1$ possible prices, $Su^i d^{N-i}$ $(i = 0, \cdots, N)$. It is convenient to display the possible paths followed by the stock price as a binomial tree [see diagram], with time going left to right and two paths, up and down, leaving each node in the tree, until we reach the $N + 1$ terminal nodes at expiry. There are 2^N possible paths through the tree. It is common to take N of the order of 30, for two reasons:

(i) typical lengths of time to expiry are measured in months (9 months, say); this gives a time-step around the corresponding number of days,

(ii) 2^{30} paths is about the order of magnitude that can be easily handled by computers (recall that $2^{10} = 1,024$, so 2^{30} is somewhat over a billion).

We now return to our treatment of the binomial model in §§5,6, with a slight change of notation. Recall that in §5 (discrete time) we used $1 + r$ for the discount factor. It is convenient to call this $1 + \rho$ instead, freeing r for its usual use as the short rate of interest in continuous time. Thus $1 + \rho = e^{r\Delta t}$, and the risk-neutrality condition $p^* = (b - r)/(b - a)$ of §5 becomes

$$p^* = (u - e^{r\Delta t})/(u - d).$$

Now recall (§7) $(1+a)/(1+r) = \exp(-\sigma/\sqrt{N})$, $(1+b)/(1+r) = \exp(\sigma/\sqrt{N})$. We replaced σ^2 by $\sigma^2 T$ (to make σ the volatility per unit time), and $T = N.\Delta t$, so $\sigma/\sqrt{N}$ becomes $\sigma\sqrt{T}/\sqrt{N} = \sigma\sqrt{\Delta t}$. So now

$$u/e^{r\Delta t} = e^{\sigma/\sqrt{\Delta t}}, \qquad d/e^{r\Delta t} = e^{-\sigma\sqrt{\Delta t}}.$$

Thus $ud = e^{2r\Delta t}$. Since $\sqrt{\Delta t}$ is small, its square Δt is a second-order term; to first order, we thus have $ud = 1$, which simplifies filling in the terminal values in the binary tree.

With an eye on this simplification, we begin again: define our up and down factors u, d so that

$$ud = 1;$$

define the risk-neutral probability p^* so as to have

$$p^* = (u - e^{r\Delta t})/(u - d)$$

(to get the mean return from the risky stock the same as that from the riskless bank account), and the volatility σ to get the variance of the stock price S' after one time-step when it is worth S initially as $S^2\sigma^2\Delta t$:

$$S^2\sigma^2\Delta t = p^* S^2 u^2 + (1-p^*)S^2 d^2 - S^2[p^* u + (1-p^*)d]^2$$

(using $var S' = E(S'^2) - [ES']^2$). Then to first order in $\sqrt{\Delta t}$ (which is all the accuracy we shall need), one can check that we have as before

$$u = \exp(\sigma\sqrt{\Delta t}), \qquad d = \exp(-\sigma\sqrt{\Delta t}).$$

We can now calculate both the value of an American put option and the optimal exercise strategy by working backwards through the tree (this method of backward recursion in time is a form of the Dynamic Programming [DP] technique (Richard Bellman (1920-84) in 1953, book, 1957), which is important in many areas of optimization and Operational Research (OR)).
1. Draw a binary tree showing the initial stock value and having the right number, N, of time-intervals.
2. Fill in the stock prices: after one time interval, these are Su (upper) and Sd (lower); after two time-intervals, Su^2, S and $Sd^2 = S/u^2$; after i time-intervals, these are $Su^j d^{i-j} = Su^{2j-i}$ at the node with j 'up' steps and $i-j$ 'down' steps (the '(i,j)' node).
3. Using the strike price K and the prices at the *terminal nodes*, fill in the

payoffs ($f_{N,j} = \max[K - Su^j d^{N-j}, 0]$) from the option at the terminal nodes (where, at expiry, the values of the European and American options coincide) underneath the terminal prices.
4. Work back down the tree one time-step. Fill in the 'European' value at the penultimate nodes as the discounted values of the upper and lower right (terminal node) values, under the risk-neutral measure - 'p^* times lower right plus $1 - p^*$ times upper right' [notation of IV.6 L18]. Fill in the 'intrinsic' (or early-exercise) value - the value if the option is exercised. Fill in the American put value as the higher of these.
5. Treat these values as 'terminal node values', and fill in the values one time-step earlier by repeating Step 4 for this 'reduced tree'.
6. Iterate. The value of the American put at time 0 is the value at the root - the last node to be filled in. The 'early-exercise region' is the node set where the early-exercise value is the higher; the rest is the 'continuation region'.
Note. The above procedure is simple to describe and understand, and simple to programme. It is laborious to implement numerically by hand, on examples big enough to be non-trivial. Numerical examples are worked through in detail in [H1], 359-360 and [CR], 241-242.

Mathematically, the task remains of describing the *continuation region* - the part of the tree where early exercise is not optimal. This is a classical *optimal stopping problem.* No explicit solution is known (and presumably there isn't one). We will, however, connect the work above with that of III.7 [L13] on the Snell envelope. Consider the pricing of an American put, strike price K, expiry N, in discrete time, with discount factor $1 + r$ per unit time as earlier. Let $Z = (Z_n)_{n=0}^N$ be the payoff on exercising at time n. We want to price Z_n, by U_n say (to conform to our earlier notation), so as to avoid arbitrage; again, we work backwards in time. The recursive step is

$$U_{n-1} = \max(Z_{n-1}, \frac{1}{1+r} E^*[U_n | \mathcal{F}_{n-1}]),$$

the first alternative on the right corresponding to early exercise, the second to the discounted expectation under P^*, as usual. Let $\tilde{U}_n = U_n/(1+r)^n$ be the discounted price of the American option. Then

$$\tilde{U}_{n-1} = \max(\tilde{Z}_{n-1}, E^*[\tilde{U}_n | \mathcal{F}_{n-1}]) :$$

$(\tilde{U}_n)$ is the *Snell envelope* (III.7) of the discounted payoff process $(\tilde{Z}_n)$. So:
(i) a P^*-supermartingale,

(ii) the smallest supermartingale dominating $(\tilde{Z}_n)$,
(iii) the solution of the optimal stopping problem for $\tilde{Z}$.
Note. One can use the Snell envelope to prove Merton's theorem (equivalence of American and European calls) without using arbitrage arguments. For details see e.g. [BK, Th. 4.7.1 and Cor. 4.7.1].
P-measure and P^*- (or $Q-$) measure.

We use P and P^* in the above, as E and E^* are convenient, but P and Q when the emphasis is on Q, for brevity.

The measure P, the *real* (or real-world) probability measure, models the uncertainty driving prices, which are indeed uncertain, thus allowing us to bring mathematics to bear on financial problems. But P is difficult to get at directly. By contrast, Q is more accessible: the *market* tells us about Q, or more specifically, *trading* does. In addition, trading also tells us about the *volatility* σ, via implied volatility, which we can infer from observing the prices at which options are traded. So Q is certainly more accessible than P. There is thus a sense in which it is Q, rather than P, which is the more real.

It is as well to bear all this in mind when looking at specific problems, particularly numerical ones. Now that we know the CRR binomial-tree model, which gives us the Black-Scholes formula in discrete time (and hence also, by the limiting argument above, the Black- Scholes formula in continuous time, the main result of the course), we can recognise the 'one-period, up or down' model ($/SFr in I.8 L5, price of gold in Problems 5), though clearly artificial and stylised, as a workable 'building block' of the whole theory. Because P itself does not occur in the Black-Scholes formula(e), there is little practical value in trying to construct more realistic, and so more complicated, models of P. In practice, one exploits instead what one can infer about Q, which does occur in Black-Scholes, from seeing the prices at which options trade.
Where we are.

The course splits neatly into three parts: Ch. I, II [L 1-10] on background, Ch. III, IV [L 11-20] on discrete time, and Ch. V, VI [L 21-30] on continuous time. We have already seen the main ideas – and proved nearly everything seen so far. In V, VI we gain the tremendous power of Itô (stochastic) calculus (calculus is our most powerful weapon, in mathematics and science!), and the ability to work in continuous time. What we lose is the ability to prove so much and to see what is happening so clearly and so concretely.