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Lecture 2. 14.10.2014

Utility (continued).

A guiding principle that is often used here is that each economic agent should seek to *maximize his expected utility.* This approach goes back to John Von Neumann and Oscar Morgenstern in 1947 (in their classic book *Theory of games and economic behaviour*, and earlier to F. P. Ramsey (1906-1930) in 1931 (posthumously).

Loss.

This is often looked at the other way round. One uses a *loss function* – which can usually be thought of as a negative of utility. One then seeks to *minimize one's expected loss.*

Arbitrage.

An *arbitrage* opportunity (see I.6) is the possibility of extracting riskless profit from the market. In an orderly market, this should not be possible – at least, to a first approximation. For, an arbitrage opportunity is 'free money'; arbitrageurs will take this, in unlimited quantities – until the person or institution being so exploited is driven from the market (bankrupt or otherwise). In view of this, we make the assumption that the market is *free of arbitrage* – is *arbitrage-free*, or has *no arbitrage*, NA.

Idealized markets.

Various assumptions are commonly made, in order to bring to bear the tools of mathematics on the broad field of economic/financial activity. All are useful, but valid to a first approximation only.

1. No arbitrage (NA).
2. No transaction costs or transaction taxes.
3. Same interest rates for borrowing and lending.
4. Unlimited liquidity (the ability to turn goods into money, and vice versa, at the currently quoted prices).
5. No limitations of scale.

Markets satisfying such assumptions will be called *perfect*, or *frictionless* – unrealistic in detail, but a useful first approximation in practice.

3. Brief history of Mathematical Finance

Mathematical Finance I: Markowitz and CAPM.

We deal with the history of put-call parity (I.7) below. It has ancient roots, but entered the textbooks around 1904.

Louis Bachelier (1870-1946) first put mathematics to work on finance in his 1900 thesis *Théorie de la spéculation*.[1]

Bachelier's thesis is also remarkable as he used *Brownian motion* as a model for the driving noise in the price of a risky asset. This was remarkable, as the relevant mathematics did not exist until 1923 (Wiener), and later (Itô, stochastic calculus, 1944).

Until 1952, finance was more an art than a science. This changed with the 1952 thesis of Harry Markowitz (1927–), which introduced modern *portfolio theory*. Markowitz gave us two key insights, both so 'obvious' that they are all around us now.

There is no point in investing in the stock market, which is risky, when one can instead invest risklessly by putting money in the bank, unless one expects the (rate of) *return* on the stock, μ, to be higher than the riskless return r. The riskiness of the stock is measured by a parameter, the *volatility* σ, which corresponds to the standard deviation (square root of the variance) in a model of the risky stock price as a stochastic process (Ch. III), while μ, r correspond to *means*, for risky and riskless assets respectively. Markowitz's first key insight is:

think of risk and return together, not separately.

This leads to *mean-variance analysis*.

Next, the investor is free to choose which sector of the economy to invest in. He is investing in the face of uncertainty (or risk), and in each sector he chooses, prices may move against him. He should insure against this by holding a *balanced portfolio*, of assets from a number of different sectors, chosen so that they will tend to 'move against each other'. Then, 'what he loses on the swings he will gain on the roundabouts'. This tendency to move against each other is measured by *negative correlation* (the term comes from Statistics). Markowitz's second key insight is:

diversify, by holding a balanced portfolio with lots of negative correlation.

Markowitz's theory was developed during the 1960s, in the *capital asset pricing model* (CAPM – 'cap-emm'), of Sharpe, Lintner and Mossin (William Sharpe (1964), John Lintner (1965), Jan Mossin (1966); Jack Treynor (1961, 1962)). In CAPM, one looks at the excess of a particular stock over that of the market overall, and the risk (as measured by volatility), and seeks to

[1] Mark Davis and Alison Etheridge: Louis Bachelier's *Theory of speculation*: The origins of modern finance, translated and with a commentary; foreword by Paul A. Samuelson. Princeton UP, 2006.

obtain the maximum return for a given risk (or minimum risk for a given return), which will hold on the *efficient frontier*. The relevant mathematics involves Linear Regression in Statistics, and Linear Programming in Operational Research (OR).

Mathematical Finance II: Black, Scholes and Merton.

If one is contemplating buying a particular stock, intending to hold it for a year say, what one would love to know is the price in a year's time, compared with the price today (one should discount this, as above). If the (discounted) price goes up, one will be glad in a year's time that one bought; if it goes down, one will be sorry.

Suppose one's Fairy Godmother appeared, and gave one a piece of paper, which said that if one bought now, then in a year's time if one was glad one had done so one did buy, but if one was sorry, one didn't. Such pieces of paper do exist, and are called *options* – see Ch. IV, VI. Clearly such options are valuable: they may lead to a profit, but cannot lead to a loss.

Question: What is an option worth?

Note that unless one can *price* options, they will not be traded (at least in any quantity) – as with anything else.

Before 1973, the conventional wisdom was that this question had no answer: it *could have no answer*, because the answer would necessarily depend on the economic agent's attitude to risk (that is, on his utility function, or loss function – see above). It turns out that this view is incorrect. Subject to the above assumptions of an idealized market (NA, etc.), one can price options, according to the famous *Black-Scholes formula* of 1973 (Ch. IV, VI – Fischer Black (1938-1995) and Myron Scholes (1941-)). They derived their formula by showing that the option price satisfied a partial differential equation (PDE), of hyperbolic type (a variant of the *heat equation*). In 1973 Robert Merton (1944-) gave a more direct approach. Meanwhile, 1973 was also the year when the first exchange for buying and selling options opened, the Chicago Board Options Exchange.

To see why options can be priced, one only needs to know that the standard options are (under our idealized assumptions) *redundant* financial assets: an option is equivalent to an appropriate combination of cash and stock. Knowing how much cash, how much stock and the current stock price, one can thus calculate the current option price by simple arithmetic.

In 1981, it was shown (by J. M. Harrison and S. R. Pliska) that the right mathematical machinery to use in this area involves a particular type of stochastic process – *martingales* – and a particular type of calculus, for stochastic processes – *Itô calculus* (Kiyosi Itô (1915-2008)); see Ch. VI.

The subject of Mathematical Finance is by now well-established, and rapidly growing in popularity in universities, in UK, US and elsewhere. This is because of its relevance to the needs of the financial sector (or financial services industry) in the City of London (also Edinburgh) within UK, New York in USA, Tokyo in Japan, Frankfurt in Germany, etc. This sector needs technical people with good skills in mathematics, statistics, numerics etc., as well as economic insight and financial awareness, problem-solving skills and ability to work in a team, etc. Such people are variously called financial engineers, quantitative analysts ('quants') or 'rocket scientists'.

Academically, the subject falls broadly in the interface between Economics on the one hand and Mathematics on the other. In Economics, much of the subject, again broadly speaking, relates to *how prices are determined* – by the interplay between supply and demand, etc. By contrast, here in this course we will usually take prices as given. Our task is to study how, starting from the given prices, we can price other things related to them (options, and other financial derivatives – see below), and guard our operations against unpredictable hazards (hedge – again, see below).

In this sense, Finance as a subject appears as a small – specialised, highly mathematical – part of Economics (note that Finance here is not used quite in the traditional non-technical sense). *Risk* is the key danger – the key concept even – in finance; risk reflects uncertainty; uncertainty reflects chance or probability. So it was clear that Probability Theory, a branch of Mathematics related to Statistics, had to be relevant here. Quite how was shown in 1981 by J. M. (Michael) Harrison (a probabilist) and David Kreps (an economist), who simplified and generalized the Black-Scholes-Merton theory by using the language of Probability Theory and Stochastic Processes – in particular, *martingales* (and Itô calculus, again). These developments – and what followed – constituted the 'second revolution in mathematical finance'. This is the subject-matter of this course. (We can cover the mathematics of the developments outlined above. More recent developments are very important, but go beyond a first undergraduate course.) On the mathematical side: you will learn a lot about stochastic processes, martingales and Itô calculus, and see them put to use on financial problems. On the practical side: the best proof of the relevance and usefulness of these ideas is the explosive growth in volumes of trades in financial derivatives over the last thirty years, and the corresponding explosive growth in employment opportunities (and salaries!) for those who understand what is going on.

As with everything else in life, triumph and disaster can always happen, and one has to use common sense. Triumph: Scholes and Merton were awarded the Nobel Prize for Economics in 1997 (Black died in 1995, and the prize cannot be awarded posthumously). Disaster: Scholes and Merton were on the board of the hedge fund Long Term Capital Management, which ignominiously collapsed with enormous losses in 1998. Pushing a good theory too far – beyond all sensible limits – is asking for trouble, even if one invented the theory and got the Nobel Prize for it, and if one asks for trouble, one can expect to get it.

4. Markets and Options.

Markets.

This course is about the mathematics needed to model *financial markets*. These are of several types:
Stock markets [New York, London, ...], dealing in stocks/shares/equities, etc.,
Bond markets, dealing in government bonds (gilts, ...),
Currency or foreign exchange ('forex') markets,
Futures and *options markets*, dealing in financial instruments derived from the above - *financial derivatives* such as *options* of various types.

Options.

Economic activity, and trading, involves *risk*. One may have to, or choose to, make a judgement involving committing funds ('taking a position') based on prediction of the future in the presence of uncertainty. With hindsight, one might or might not regret taking that position. An *option* is a financial instrument giving one the *right but not the obligation* to make a specified transaction at (or by) a specified date at a specified price. Whether or not the option will be exercised depends on (is contingent on) the uncertain future, so is also known as a *contingent claim.*

Types of option.

Call options give one the right (but not – without further comment now – the obligation) to *buy.*
Put options give one the right to *sell.*
European options give one the right to buy/sell *on* the specified date, the *expiry* date, when the option *expires* or matures.
American options give one the right to buy/sell *at any time* prior to or at expiry. Thus:

European options: exercise *at* expiry,
American options: exercise *by* expiry.

Note. The terms European, American (Asian, Bermudan, Russian, ...) refer only to the type of option, and no longer bear any relation to the area in the name. Most options traded worldwide these days are American.

History. As discussed in §1, over-the-counter (OTC) options were long ago negotiated by a broker between a buyer and a seller. Then in 1973 (the year of the Black-Scholes formula, perhaps the central result of the course), the Chicago Board Options Exchange (CBOE) began trading in options on some stocks. Since then, the growth of options has been explosive. Options are now traded on all the major world exchanges, in enormous volumes. Often, the market in derivatives is *much larger* than the market in the underlying assets – an important source of instability in financial markets.

The simplest call and put options are now so standard they are called *vanilla* options. Many kinds of options now exist, including so-called *exotic* options. Types include:

Asian options, which depend on the *average* price over a period,

Russian options, or other *lookback* options, which depend on the *maximum* or *minimum* price over a period,

Barrier options, which depend on some price level being attained or not.

Real options (also called *investment options*). These are 'options' available to the management of a company considering whether or when to commit capital (usually both irreversibly and riskily) to some investment project. Waiting may be valuable, as one can gather more information.

Terminology. The asset to which the option refers is called the *underlying asset* or the *underlying*. The price at which the transaction to buy/sell the underlying, on/by the expiry date (if exercised), is called the *exercise price* or *strike price*. We shall usually use K for the strike price, time $t = 0$ for the initial time (when the contract between the buyer and the seller of the option is struck), time $t = T$ for the expiry or final time.

Consider, say, a European *call* option, with strike price K; write S_t for the value (or price) of the underlying at time t. If $S_T > K$, the option is *in the money*: the holder will/should *exercise* the option, obtaining an asset worth S_T ($> K$) for K. He can immediately sell the asset for S_T, making a *profit* of $S_T - K$ (> 0).

If $S_T = K$, the option is said to be *at the money*.

If $S_T < K$, the option is *out of the money*, and should not be exercised. It is worthless, and is thrown away.

The *pay-off* from the option is thus

$$S_T - K \text{ if } S_T > K, \quad 0 \quad \text{otherwise,}$$

which may be written more briefly as

$$max(S_T - K, 0) \text{ or } (S_T - K)_+$$

$(x_+ := max(x, 0), x_- := -min(x, 0); x = x_+ - x_-, |x| = x_+ + x_-)$.
Similarly, the payoff from a *put* option is

$$K - S_T \text{ if } S_T \leq K, \qquad 0 \text{ if } S_T > K,$$

or $(K - S_T)_+$.
Option pricing. The fundamental problem in the mathematics of options is that of *option pricing*. The modern theory began with the *Black-Scholes formula* for pricing European options in 1973. We shall deal with the Black-Scholes theory, and cover the pricing of European options in full. We also discuss American options: these are harder, and lack explicit formulae such as the Black-Scholes formula; consequently, one needs to evaluate them numerically. The pricing of Asian options is even harder and is still topical at research level.
Perfect Markets. For simplicity, we shall confine ourselves to option pricing in the simplest (idealised) case, of a *perfect*, or *frictionless*, market. First, there are no *transaction costs* (one can include transaction costs in the theory, but this is considerably harder). Similarly, we assume that interest rates for borrowing and for lending are the same (which is unrealistic, as banks make their money on the difference), and also that all traders have access to the same – perfect – information about the past history of price movements, but have no foreknowledge of price-sensitive information (i.e. no insider trading). We shall assume no restrictions on *liquidity* – that is, one can buy or sell unlimited quantities of stock at the currently quoted price. That is, our economic agents are *price takers* and not *price makers*. (This comes back to §1 on the relationship between Economics and Finance. In practice, big trades do move markets. Also, in a crisis, no-one wants to trade, and liquidity dries up – basically, this is what did for LTCM.) In practice, very small trades are not economic (the stockbroker may only deal in units of reasonable size, etc.). We shall ignore all these complications for the sake of simplicity.
References. For completeness, classic papers include:

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