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§7. More on European Options

1. *Bounds.*

We use the notation above. We also write c, p for the values of European calls and puts, C, P for the values of the American counterparts.

Obvious upper bounds are $c \leq S, C \leq S$, where S is the stock price (we can buy for S on the market without worrying about options, so would not pay more than this for the right to buy). For puts, one has correspondingly the obvious upper bounds $p \leq K, P \leq K$, where K is the strike price: one would not pay more than K for the right to sell at price K, as one would not spend more than one's maximum return. For lower bounds:

$c_0 \geq \max(S_0 - Ke^{-rT}, 0)$.

Proof. Consider the following two portfolios:

I: one European call plus Ke^{-rT} in cash; II: one share. Show "I $\geq$ II".

$p_0 \geq \max(Ke^{-rT} - S_0, 0)$.

Proof. By above and put-call parity.

2. *Dependence of the Black-Scholes price on the parameters.*

Recall the Black-Scholes formulae for the values c_t, p_t for the European call and put: with

$$d_\pm := [\log(S_t/K) + (r \pm \frac{1}{2}\sigma^2)(T-t)]/\sigma\sqrt{(T-t)},$$

$$c_t = S_t\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-), \qquad p_t = Ke^{-r(T-t)}\Phi(-d_-) - S_t\Phi(-d_+),$$

1. S. As the stock price S increases, the call option becomes more and more likely to be exercised. In the limit for large S, $d_\pm \to \infty$, $\Phi(d_\pm) \to 1$, so $c_t \to S_t - Ke^{-r(T-t)}$. This limit has a natural economic interpretation: it is the value of a *forward contract* with *delivery price* K (see e.g. Hull [H1] Ch. 3, [H2] Ch. 3).

2. σ. When the volatility $\sigma \to 0$, the stock becomes riskless, and behaves like money in the bank. Again, $d_\pm \to \infty$, the Black-Scholes price has the limit above, and one has the correct economic interpretation.

3. *Volatility.*

As in IV.6.6 L18, the volatility σ can be estimated in two ways:

a. Directly from the movement of a stock price in time [as the mathematics here is continuous time, we defer it to Ch. VI], giving what is called the *historic volatility.*

b. From the observed market prices of options: if we know everything in the Black-Scholes formula (including the price at which the option is traded) except the volatility σ, we can solve for σ. This is called *implied volatility.* Since σ appears inside the argument of the normal distribution function Φ as well as outside it, this is a transcendental equation for σ and has to be solved numerically by iteration (Newton-Raphson method). We quote (see 'The Greeks' below, and Problems 7) that the Black-Scholes price is a monotone (increasing) function of the volatility (more volatility doesn't make us 'more likely to win', but when we do win, we 'win bigger'), so there is a unique root of the equation.

In practice, one sees discrepancies between historic and implied volatility, which show limitations to the accuracy of the Black-Scholes model. But it is the standard 'benchmark model', and useful as a first approximation.

The classical view of volatility is that it is caused by future uncertainty, and shows the market's reaction to the stream of new information. However, studies taking into account periods when the markets are open and closed [there are only about 250 trading days in the year] have shown that the volatility is less when markets are closed than when they are open. This suggests that *trading itself is one of the main causes of volatility.*

Note. This observation has deep implications for the macro-prudential and regulatory issues discussed in Ch. 1. The real economy cannot afford too much volatility. Volatility is (at least partly) caused by trading. Conclusion: there is too much trading. Policy question: how can we reduce the volume of trading (much of it speculative, designed to enrich traders, and not serving a more widely useful economic purpose)? One answer is the so-called *Tobin tax* (also known as the "Robin Hood tax") (James Tobin (1918-2002), American economist; Nobel Prize for Economics, 1981). This would levy a small charge (e.g. 0.01%) on *all* financial transactions. This would both provide a major and useful source of tax revenue, and – more importantly – would discourage a lot of speculative trading, thereby (shrinking the size of the financial services industry, but) diminishing volatility, to the benefit of the general economy (Problems 7 again).

4. *The Greeks.*

These are the partial derivatives of the option price with respect to the input parameters. They have the interpretation of *sensitivities.*

(i) For a call, say, $\partial c/\partial S$ is called the *delta*, Δ. Adjusting our holdings of stock to eliminate our portfolio's dependence on S is called *delta-hedging.*

(ii) Second-order effects involve *gamma* $:= \partial(\Delta)/\partial S$.

(iii) Time-dependence is given by *Theta* is $\partial c/\partial t$.
(iv) Volatility dependence is given by *vega* $:= \partial c/\partial \sigma$.[1]

From the Black-Scholes formula (which gives the price explicitly as a function of σ), one can check by calculus (Problems 7) that

$$\partial c/\partial \sigma > 0,$$

and similarly for puts (or, use the result for calls and put-call parity). In sum: *options like volatility.* This fits our intuition. The more uncertain things are (the higher the volatility), the more valuable protection against adversity – or insurance against it – becomes (the higher the option price).
(v) *rho* is $\partial c/\partial r$, the sensitivity to interest rates.

§8. American Options.

We now consider an American call option (value C), in the simplest case of a stock paying no dividends. The following result goes back (at least) to R. C. MERTON in 1973.

Theorem (Merton's theorem). It is never optimal to exercise an American call option early. That is, the American call option is equivalent to the European call, so has the same value:

$$C = c.$$

First Proof. Consider the following two portfolios:
I: one American call option plus cash Ke^{-rT}; II: one share.
The value of the cash in I is K at time T, $Ke^{-r(T-t)}$ at time t. If the call option is exercised early at $t < T$, the value of Portfolio I is then $S_t - K$ from the call, $Ke^{-r(T-t)}$ from the cash, total

$$S_t - K + Ke^{-r(T-t)}.$$

Since $r > 0$ and $t < T$, this is $< S_t$, the value of Portfolio II at t. So Portfolio I is *always* worth less than Portfolio II if exercised *early.*

If however the option is exercised instead at expiry, T, the American call option is then the same as a European call option. We are then in

[1] Of course, vega is not a letter of the Greek alphabet! (it is the Spanish word for 'meadow', as in Las Vegas) – presumably so named for "v for volatility, variance and vega", and because vega sounds quite like beta, etc.

the situation of §7.1 above: at time T, Portfolio I is worth $\max(S_T, K)$ and Portfolio II is worth S_T. So:

$$\begin{array}{lc} \text{before} \quad T, & I < II, \\ \text{at} \quad T, & I \geq II \text{ always, and } > \text{sometimes.} \end{array}$$

This direct comparison with the underlying [the share in Portfolio II] shows that early exercise is never optimal. Since an American option at expiry is the same as a European one, this completes the proof. //

Second Proof. One can prove the result without arbitrage arguments by using the bounds of §7.1. For details, see e.g. [BK, Th. 4.7.1].

Financial Interpretation.

There are two reasons why an American call should not be exercised early:

1. *Insurance.* Consider an investor choosing to hold a call option instead of the underlying stock. He does not care if the share price falls below the strike price (as he can then just discard his option) – but if he held the stock, he would. Thus the option insures the investor against such a fall in stock price, and if he exercises early, he loses this insurance.

2. *Interest on the strike price.* When the holder exercises the option, he buys the stock and pays the strike price, K. Early exercise at $t < T$ loses the interest on K between times t and T: the later he pays out K, the better.

Economic Note. Despite Merton's theorem, and the interpretation above, there are plenty of real-life situations where early exercise of an American call might be sensible, and indeed done routinely. Consider, for example, a manufacturer of electrical goods, in bulk. He needs a regular supply of large amounts of copper. The danger is future price increases; the obvious precaution is to hedge against this by buying call options. If the expiry is a year but copper stocks are running low after six months, he would exercise his American call early, to keep an adequate inventory of copper, his crucial raw material. This ensures that his main business activity – manufacturing – can continue unobstructed. Neither of the reasons above applies here:

Insurance. He doesn't care if the price of copper falls: he isn't going to sell his copper stocks, but use them.

Interest. He doesn't care about losing interest on cash over the remaining six months. He is in manufacturing to use his money to make things, and then sell them, rather than put it in the bank.

This neatly illustrates the contrast between *finance* (money, options etc.) and *economics* (the real economy – goods and services).