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**Lecture 18 21.10.2014**

*Proof (continued).*
This is the characteristic function of the normal law $N(\mu, \sigma^2)$. The result follows, since convergence of CFs implies convergence in distribution by Lévy's continuity theorem for CFs ([W], §18.1). //

We can apply this to pricing the call option above:

$$C_0^{(N)} = (1 + \frac{RT}{N})^{-N} E^*[(S_0 \Pi_1^N T_n - K)_+]$$
$$= E^*[(S_0 \exp\{Y_N\} - (1 + \frac{RT}{N})^{-N} K)_+], \qquad (1)$$

where

$$Y_N := \sum\nolimits_1^N \log(T_n/(1+r)).$$

Since $T_n = T_n^N$ above takes values $1+b, 1+a$, $X_n^N := \log(T_n^N/(1+r))$ takes values $\log((1+b)/(1+r))$, $\log((1+a)/(1+r)) = \pm\sigma/\sqrt{N}$ (so has second moment $\sigma^2/N$). Its mean is

$$\mu_N := \log\Big(\frac{1+b}{1+r}\Big)(1-p^*) + \log\Big(\frac{1+a}{1+r}\Big)p^* = \frac{\sigma}{\sqrt{N}}(1-p^*) - \frac{\sigma}{\sqrt{N}}p^* = (1-2p^*)\sigma/\sqrt{N}$$

(we shall see below that $1 - 2p^* = O(1/\sqrt{N})$, so the Lemma will apply). Now (recall $r = RT/N = O(1/N)$)

$$a = (1+r)e^{-\sigma/\sqrt{N}} - 1, \qquad b = (1+r)e^{\sigma/\sqrt{N}} - 1,$$

so $a, b, r \to 0$ as $N \to \infty$, and

$$1 - 2p^* = 1 - 2\frac{(b-r)}{(b-a)} = 1 - 2\frac{[(1+r)e^{\sigma/\sqrt{N}} - 1 - r]}{[(1+r)(e^{\sigma/\sqrt{N}} - e^{-\sigma/\sqrt{N}})]}$$
$$= 1 - 2\frac{[e^{\sigma/\sqrt{N}} - 1]}{[e^{\sigma/\sqrt{N}} - e^{-\sigma/\sqrt{N}}]}.$$

Now expand the two $[\cdots]$ terms above by Taylor's theorem: they give

$$\frac{\sigma}{\sqrt{N}}(1 + \frac{1}{2}\frac{\sigma}{\sqrt{N}} + \cdots), \qquad \frac{2\sigma}{\sqrt{N}}(1 + \frac{\sigma^2}{6N} + \cdots).$$

So, cancelling $\sigma/\sqrt{N}$,

$$1 - 2p^* = 1 - \frac{2(1 + \frac{1}{2}\frac{\sigma}{\sqrt{N}} + \cdots)}{2(1 + \frac{\sigma^2}{6N} + \cdots)} = -\frac{1}{2}\frac{\sigma}{\sqrt{N}} + O(1/N):$$

$$N\mu_N = N.\frac{\sigma}{\sqrt{N}}.(-\frac{1}{2}\frac{\sigma}{\sqrt{N}} + O(1/N)) \to \mu := -\frac{1}{2}\sigma^2 \qquad (N \to \infty).$$

We are thus in the situation of the Lemma, with $\mu = -\frac{1}{2}\sigma^2$. In (1), we have $Y_N \to Y$ in distribution and $(1 + \frac{RT}{N})^{-N} \to e^{-RT}$ as $N \to \infty$. This suggests that

$$C_0^{(N)} \to E[(S_0 e^Y - e^{-RT}K)_+],$$

where $E$ is the expectation for the distribution of $Y$, which is $N(-\frac{1}{2}\sigma^2, \sigma^2)$. This can be justified, by standard properties of convergence in distribution (see e.g. [W]). So if $Z := (Y + \frac{1}{2}\sigma^2)/\sigma$, $Z \sim N(0,1)$, $Y = -\frac{1}{2}\sigma^2 + \sigma Z$, and

$$C_0^{(N)} \to \int_{-\infty}^{\infty} [S_0 \exp\{-\frac{1}{2}\sigma^2 + \sigma x\} - e^{-RT}K]_+ \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} dx \qquad (N \to \infty).$$

To evaluate the integral, note first that $[...] > 0$ where

$$S_0 \exp\{-\frac{1}{2}\sigma^2 + \sigma x\} > e^{-RT}K, \qquad -\frac{1}{2}\sigma^2 + \sigma x > \log(K/S_0) - RT:$$

$$x > [\log(K/S_0) + \frac{1}{2}\sigma^2 - RT]/\sigma = c, \quad \text{say.}$$

So writing $\Phi(x)$ for the standard normal distribution function,

$$C_0 = S_0 \int_c^{\infty} e^{-\frac{1}{2}\sigma^2}.\exp\{-\frac{1}{2}x^2 + \sigma x\} dx/\sqrt{2\pi} - Ke^{-RT}[1 - \Phi(c)].$$

The remaining integral is

$$\int_c^{\infty} \exp\{-\frac{1}{2}(x-\sigma)^2\} dx/\sqrt{2\pi} = \int_{c-\sigma}^{\infty} \exp\{-\frac{1}{2}u^2\} du/\sqrt{2\pi} = 1 - \Phi(c - \sigma).$$

So the option price is given as a function of the initial price $S_0$, strike price $K$, expiry $T$, interest rate $R$ and variance $\sigma^2$ by

$$C_0 = S_0[1 - \Phi(c - \sigma)] - Ke^{-RT}[1 - \Phi(c)], \qquad c = [\log(K/S_0) + \frac{1}{2}\sigma^2 - RT]/\sigma.$$

To compare with our later work, it is convenient now to replace $\sigma^2$ by $\sigma^2 T$; thus $\sigma^2$ is now the variance per unit time. Its square root, $\sigma$, is called the *volatility* of the stock. Then $c - \sigma$, $c$ above become $c_\pm$, where

$$c_\pm := [\log(K/S_0) - (R \pm \frac{1}{2}\sigma^2)T]/\sigma\sqrt{T}.$$

The result extends immediately to give the price of the option at time $t \in (0, T)$, by replacing $T$ by $T - t$, $S_0$ by $S_t$.

We re-write the formula in more customary notation. First, write $r$ in place of $R$ for the interest rate. Next, using the symmetry of the normal distribution, $1 - \Phi(c_\pm) = \Phi(-c_\pm) = \Phi(d_\pm)$, say, where

$$d_\pm := -c_\pm = [\log(S/K) + (r \pm \frac{1}{2}\sigma^2)(T - t)]/\sigma\sqrt{T - t}:$$

the price of the European call option is

$$c_t = S_t \Phi(d_+) - e^{-r(T-t)} K \Phi(d_-).$$

This is the famous *continuous Black-Scholes formula*. We shall return to it in Chapter VI, where we re-derive it by continuous-time methods (Brownian motion and Itô calculus).

*Note.* 1. The same argument (or put-call parity) gives the value of the European put option as $p_t = Ke^{-r(T-t)}\Phi(-d_-) - S_t\Phi(-d_+)$.

2. The proof above starts from a binomial distribution and ends with a normal distribution. The binomial distribution is that of a sum of independent Bernoulli random variables. That sums (equivalently, averages) of independent random variables with finite means and variances gives a normal limit is the content of the Central Limit Theorem or CLT (the *Law of Errors*, as physicists would say). The particular form of the CLT used here – normal approximation to the binomial – is the *de Moivre-Laplace limit theorem.*

The picture for this is familiar. The Binomial distribution $B(n, p)$ has a histogram with $n + 1$ bars, whose heights peak at the mode and decrease to either side. For large $n$, one can draw a smooth curve through the histogram. The curve looks like a normal density curve (with the appropriate location and scale, i.e. mean and variance). The result proved above, and the classical de Moivre-Laplace limit theorem, say that this is exactly right.

3. The Cox-Ross-Rubinstein binomial model above goes over in the passage to the limit to the geometric Brownian motion model of VI.1. We will later

re-derive the continuous Black-Scholes formula in Ch. VI, using continuous-time methods (Itô calculus), rather than using the method above of deriving the discrete Black-Scholes formula and going to the limit on the *formula*, rather than the *model*.
4. For similar derivations of the discrete Black-Scholes formula and the passage to the limit to the continuous Black-Scholes formula, see e.g. [CR], §5.6.
5. One of the most striking features of the Black-Scholes formula is that it does **not** involve the mean rate of return $\mu$ of the stock - only the riskless interest-rate $r$ and the volatility of the stock $\sigma$. Mathematically, this reflects the fact that the change of measure involved in the passage to the risk-neutral measure involves a change of drift. This eliminates the $\mu$ term; see Ch. VI.
6. The Black-Scholes formula involves the parameter $\sigma$ (where $\sigma^2$ is the variance of the stock per unit time), called the *volatility* of the stock. In financial terms, this represents how sensitive the stock-price is to new information – how 'volatile' the market's assessment of the stock is. This volatility parameter is very important, *but* we do not know it; instead, we have to *estimate* the volatility for ourselves. There are two approaches:
(a) *historic volatility*: here we use Time Series methods to estimate $\sigma$ from past price data. Clearly the more variability we observe in runs of past prices, the more volatile the stock price is, and given enough data we can estimate $\sigma$ in this way.
(b) *implied volatility*: match observed option prices to theoretical option prices. For, the price we see options traded at tells us what the *market* thinks the volatility is (estimating volatility this way works because the dependence is monotone; see later).

If the Black-Scholes model were perfect, these two estimates would agree (to within sampling error). But discrepancies can be observed, which shows the imperfections of our model.

Volatility estimation is a major topic, both theoretically and in practice. We return to this in IV.7.3-4 below and VI.7.5-8. But looking ahead:
(i) trading is itself one of the major causes of volatility;
(ii) options like volatility [i.e., option prices go up with volatility].
Recalling Ch. I, this shows that volatility is a 'bad thing' from the point of view of the real economy (uncertainty about, e.g., future material costs is nothing but a nuisance to manufacturers), but a 'good thing' for financial markets (trading increases volatility, which increases option prices, which generates more trade ...) – at the cost of increased instability.