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We note that, to calculate prices as above, we need to know only
(i) Ω, the set of all possible states,
(ii) the σ-field $\mathcal{F}$ and the filtration (or information flow) $(\mathcal{F}_n)$,
(iii) the EMM P^* (or Q).
We do **NOT** need to know the underlying probability measure P – only its null sets, to know what 'equivalent to P' means (actually, in this model, only the empty set is null).

Now option pricing is our central task, and for pricing purposes P^* is vital and P itself irrelevant. We thus may – and shall – focus attention on P^*, which is called the *risk-neutral* probability measure. *Risk-neutrality* is the central concept of the subject. The concept of risk-neutrality is due in its modern form to Harrison and Pliska [HP] in 1981 – though the idea can be traced back to actuarial practice much earlier. Harrison and Pliska call P^* the *reference measure*; other names are *risk-adjusted* or *martingale measure*. The term 'risk-neutral' reflects the P^*-martingale property of the risky assets, since martingales model fair games.

To summarise, we have the

Theorem (Risk-Neutral Pricing Formula). In a complete viable market, arbitrage-free prices of assets are their discounted expected values under the risk-neutral (equivalent martingale) measure P^* (or Q). With payoff h,

$$V_n(H) = (1+r)^{-(N-n)} E^*[V_N(H)|\mathcal{F}_n] = (1+r)^{-(N-n)} E^*[h|\mathcal{F}_n].$$

§5. European Options. The Discrete Black-Scholes Formula.

We consider the simplest case, the Cox-Ross-Rubinstein *binomial model* of 1979; see [CR], [BK]. We take $d = 1$ for simplicity (one risky asset, one riskless asset or bank account); the price vector is (S_n^0, S_n^1), or $((1+r)^n, S_n)$, where

$$S_{n+1} = \begin{cases} S_n(1+a) & \text{with probability } p, \\ S_n(1+b) & \text{with probability } 1-p \end{cases}$$

with $-1 < a < b$, $S_0 > 0$. So writing N for the expiry time,

$$\Omega = \{1+a, 1+b\}^N,$$

each $\omega \in \Omega$ representing the successive values of $T_{n+1} := S_{n+1}/S_n$, $n = 0, 1, \cdots, N-1$. The filtration is $\mathcal{F}_0 = \{\emptyset, \Omega\}$ (trivial σ-field), $\mathcal{F}_T = \mathcal{F} = 2^\Omega$ (power-set of Ω: class of all subsets of Ω), $\mathcal{F}_n = \sigma(S_1, \cdots, S_n) = \sigma(T_1, \cdots, T_n)$. For $\omega = (\omega_1, \cdots, \omega_N) \in \Omega$, $P(\{\omega_1, \cdots, \omega_N\}) = P(T_1 = \omega_1, \cdots, T_N = \omega_N)$, so knowing the probability measure P (equivalently, knowing p) means we know the distribution of $(T_1, \cdots, T_N)$.

For $p^* \in (0, 1)$ to be determined, let P^* correspond to p^* as P does to p. Then the discounted price $(\tilde{S}_n)$ is a P^*-martingale iff

$$E^*[\tilde{S}_{n+1}|\mathcal{F}_n] = \tilde{S}_n, \qquad E^*[(\tilde{S}_{n+1}/\tilde{S}_n)|\mathcal{F}_n] = 1,$$

$$E^*[T_{n+1}|\mathcal{F}_n] = 1 + r \qquad (n = 0, 1, \cdots, N-1),$$

since $S_n = \tilde{S}_n(1+r)^n$, $T_{n+1} = S_{n+1}/S_n = (\tilde{S}_{n+1}/\tilde{S}_n)(1+r)$. But

$$E^*(T_{n+1}|\mathcal{F}_n) = (1+a).p^* + (1+b).(1-p^*)$$

is a weighted average of $1+a$ and $1+b$; this can be $1+r$ iff $r \in [a, b]$. As P^* is to be *equivalent* to P and P has no non-empty null-sets, $r = a, b$ are excluded. Thus by §2:

Lemma. The market is viable (arbitrage-free) iff $r \in (a, b)$.

Next, $1+r = (1+a)p^* + (1+b)(1-p^*)$, $r = ap^* + b(1-p^*)$: $r - b = p^*(a-b)$:

Lemma. The equivalent martingale measure exists, is unique, and is given by

$$p^* = (b-r)/(b-a).$$

Corollary. The market is complete.

Now $S_N = S_n \Pi_{n+1}^N T_i$. By the Fundamental Theorem of Asset Pricing, the price C_n of a call option with strike-price K at time n is

$$\begin{aligned} C_n &= (1+r)^{-(N-n)} E^*[(S_N - K)_+|\mathcal{F}_n] \\ &= (1+r)^{-(N-n)} E^*[(S_n \Pi_{n+1}^N T_i - K)_+|\mathcal{F}_n]. \end{aligned}$$

Now the conditioning on $\mathcal{F}_n$ has no effect – on S_n as this is $\mathcal{F}_n$-measurable (known at time n), and on the T_i as these are independent of $\mathcal{F}_n$. So

$$\begin{aligned} C_n &= (1+r)^{-(N-n)} E^*[(S_n \Pi_{n+1}^N T_i - K)_+] \\ &= (1+r)^{-(N-n)} \Sigma_{j=0}^{N-n} \binom{N-n}{j} p^{*j} (1-p^*)^{N-n-j} (S_n(1+a)^j (1+b)^{N-n-j} - K)_+; \end{aligned}$$

here j, $N-n-j$ are the numbers of times T_i takes the two possible values $1+a, 1+b$. This is the *discrete Black-Scholes formula* of Cox, Ross & Rubinstein (1979) for pricing a European call option in the binomial model. The European put is similar – or use put-call parity (I.3).

To find the (perfect-hedge) strategy for replicating this explicitly: write

$$c(n,x) := \Sigma_{j=0}^{N-n} \binom{N-n}{j} p^{*j}(1-p^*)^{N-n-j}(x(1+a)^j(1+b)^{N-n-j} - K)_+.$$

Then $c(n,x)$ is the undiscounted P^*-expectation of the call at time n given that $S_n = x$. This must be the value of the portfolio at time n if the strategy $H = (H_n)$ replicates the claim:

$$H_n^0(1+r)^n + H_n S_n = c(n, S_n)$$

(here by previsibility H_n^0 and H_n are both functions of $S_0, \cdots, S_{n-1}$ only). Now $S_n = S_{n-1}T_n = S_{n-1}(1+a)$ or $S_{n-1}(1+b)$, so:

$$\begin{aligned} H_n^0(1+r)^n + H_n S_{n-1}(1+a) &= c(n, S_{n-1}(1+a)) \\ H_n^0(1+r)^n + H_n S_{n-1}(1+b) &= c(n, S_{n-1}(1+b)). \end{aligned}$$

Subtract:

$$H_n S_{n-1}(b-a) = c(n, S_{n-1}(1+b)) - c(n, S_{n-1}(1+a)).$$

So H_n in fact depends only on $S_{n-1}, H_n = H_n(S_{n-1})$ (by previsibility), and

Proposition. The perfect hedging strategy H_n replicating the European call option above is given by

$$H_n = H_n(S_{n-1}) = \frac{c(n, S_{n-1}(1+b)) - c(n, S_{n-1}(1+a))}{S_{n-1}(b-a)}.$$

Notice that the numerator is the difference of two values of $c(n,x)$ with the larger value of x in the first term (recall $b > a$). When the payoff function $c(n,x)$ is an increasing function of x, as for the European call option considered here, this is non-negative. In this case, the Proposition gives $H_n \geq 0$: the replicating strategy does not involve short-selling. We record this as:

Corollary. When the payoff function is a non-decreasing function of the final asset price S_N, the perfect-hedging strategy replicating the claim does not involve short-selling of the risky asset.

§6. Continuous-Time Limit of the Binomial Model.

Suppose now that we wish to price an option in continuous time with initial stock price S_0, strike price K and expiry T. We can use the work above to give a discrete-time approximation, where $N \to \infty$. Given $R \geq 0$, the instantaneous interest rate in continuous time, define r by

$$r := RT/N : \qquad e^{RT} = \lim_{N\to\infty}(1 + \frac{RT}{N})^N = \lim_{N\to\infty}(1+r)^N.$$

Here r, which tends to zero as $N \to \infty$, represents the interest rate in discrete time for the approximating binomial model.
For $\sigma > 0$ fixed (σ^2 is to be a variance in continuous time, which will correspond to the *volatility* of the stock), define a, b by

$$\log((1+a)/(1+r)) = -\sigma/\sqrt{N}, \qquad \log((1+b)/(1+r)) = \sigma/\sqrt{N}$$

(a, b both go to zero as $N \to \infty$). We now have a sequence of binomial models, for each of which we can price options as in §5. We shall show that the pricing formula converges as $N \to \infty$ to a limit (we identify this later with the continuous Black-Scholes formula of Ch. VI); see e.g. [BK], 4.6.2.

Lemma. Let $(X_j^N)_{j=1}^N$ be iid with mean μ_N satisfying

$$N\mu_N \to \mu \qquad (N \to \infty)$$

and variance $\sigma^2(1+o(1))/N$. If $Y_N := \Sigma_1^N X_j^N$, then Y_N converges in distribution to normality:

$$Y_N \to Y = N(\mu, \sigma^2) \qquad (N \to \infty).$$

Proof. Use characteristic functions: since Y_N has mean $\mu_N = \mu(1+o(1))/N$ and variance as given, it also has second moment $\sigma^2(1+o(1))/N$. So it has characteristic function (CF)

$$\phi_N(u) := E\exp\{iuY_N\} = \Pi_1^N E\exp\{iuX_j^N\} = [E\exp\{iuX_1^N\}]^N$$

$$= (1 + \frac{iu\mu}{N} - \frac{1}{2}\frac{\sigma^2 u^2}{N} + o(\frac{1}{N}))^N \to \exp\{iu\mu - \frac{1}{2}\sigma^2 u^2\} \qquad (N \to \infty).$$