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§3. Complete Markets: Uniqueness of EMMs.

A contingent claim (option, etc.) can be defined by its *payoff* function, h say, which should be non-negative (options confer rights, not obligations, so negative values are avoided by not exercising the option), and $\mathcal{F}_N$-measurable (so that we know how to evaluate h at the terminal time N).

Definition. A contingent claim defined by the payoff function h is *attainable* if there is an admissible strategy worth (i.e., replicating) h at time N. A market is *complete* if every contingent claim is attainable.

Theorem (complete iff EMM unique). A viable market is complete iff there exists a unique probability measure P^* equivalent to P under which discounted asset prices are martingales – that is, iff equivalent martingale measures are unique.

Proof. $\Rightarrow$: Assume viability and completeness. Then for any $\mathcal{F}_N$-measurable random variable $h \geq 0$, there exists an admissible (so self-financing) strategy H replicating h: $h = V_N(H)$. As H is self-financing, by §1

$$h/S_N^0 = \tilde{V}_N(H) = V_0(H) + \Sigma_1^N H_j.\Delta\tilde{S}_j.$$

We know by the Theorem of §2 that an equivalent martingale measure P^* exists; we have to prove uniqueness. So, let P_1, P_2 be two such equivalent martingale measures. For $i = 1, 2$, $(\tilde{V}_n(H))_{n=0}^N$ is a P_i-martingale. So,

$$E_i(\tilde{V}_N(H)) = E_i(V_0(H)) = V_0(H),$$

since the value at time zero is non-random ($\mathcal{F}_0 = \{\emptyset, \Omega\}$). So

$$E_1(h/S_N^0) = E_2(h/S_N^0).$$

Since h is arbitrary, E_1, E_2 have to agree on integrating all non-negative integrands. Taking negatives and using linearity: they have to agree on non-positive integrands also. Splitting an arbitrary integrand into its positive and negative parts: they have to agree on all integrands. Now E_i is expectation (i.e., integration) with respect to the measure P_i, and measures that agree on integrating all integrands must coincide. So $P_1 = P_2$. //

Before proving the converse, we prove a lemma. Recall that an admissible strategy is a self-financing strategy with all values non-negative. The Lemma shows that the non-negativity of contingent claims extends to all values of any self-financing strategy replicating it – in other words, this gives equivalence of admissible and self-financing replicating strategies.

Lemma. In a viable market, any attainable h (i.e., any h that can be replicated by an admissible strategy H) can also be replicated by a self-financing strategy H.

Proof. If H is self-financing and P^* is an equivalent martingale measure under which discounted prices $\tilde{S}$ are P^*-martingales (such P^* exist by viability and the Theorem of §2), $\tilde{V}_n(H)$ is also a P^*-martingale, being the martingale transform of $\tilde{S}$ by H (see §1). So

$$\tilde{V}_n(H) = E^*(\tilde{V}_N(H)|\mathcal{F}_n) \qquad (n = 0, 1, \cdots, N).$$

If H replicates h, $V_N(H) = h \geq 0$, so discounting, $\tilde{V}_N(H) \geq 0$, so the above equation gives $\tilde{V}_n(H) \geq 0$ for each n. Thus *all* the values at each time n are non-negative – not just the final value at time N – so H is admissible. //

Proof of the Theorem (continued). $\Leftarrow$ (*not examinable)*: Assume the market is viable but incomplete: then there exists a non-attainable $h \geq 0$. By the Lemma, we may confine attention to self-financing strategies H (which will then automatically be admissible). By the Proposition of §1, we may confine attention to the risky assets $S^1, \cdots, S^d$, as these suffice to tell us how to handle the bank account S^0.

Call $\tilde{\mathcal{V}}$ the set of random variables of the form

$$U_0 + \Sigma_1^N H_n . \Delta \tilde{S}_n$$

with U_0 $\mathcal{F}_0$-measurable (i.e. deterministic) and $((H_n^1, \cdots, H_n^d))_{n=0}^N$ predictable; this is a vector space. Then by above, the discounted value h/S_N^0 does not belong to $\tilde{\mathcal{V}}$, so $\tilde{\mathcal{V}}$ is a *proper* subspace of the vector space $\mathbb{R}^\Omega$ of all random variables on Ω. Let P^* be a probability measure equivalent to P under which discounted prices are martingales (such P^* exist by viability, by the Theorem of §2). Define the scalar product

$$(X, Y) \to E^*(XY)$$

on random variables on Ω. Since $\tilde{\mathcal{V}}$ is a proper subspace, by Gram-Schmidt orthogonalisation there exists a non-zero random variable X orthogonal to $\tilde{\mathcal{V}}$. That is,

$$E^*[X] = 0.$$

Write $\|X\|_\infty := \max\{|X(\omega)| : \omega \in \Omega\}$, and define P^{**} by

$$P^{**}(\{\omega\}) = \Big(1 + \frac{X(\omega)}{2\|X\|_\infty}\Big) P^*(\{\omega\}).$$

By construction, P^{**} is equivalent to P^* (same null-sets - actually, as $P^* \sim P$ and P has no non-empty null-sets, neither do P^*, P^{**}). As X is non-zero, P^{**} and P^* are *different*. Now

$$\begin{aligned} E^{**}[\Sigma_1^N H_n.\Delta\tilde{S}_n] &= \Sigma_\omega P^{**}(\omega)\Big(\Sigma_1^N H_n.\Delta\tilde{S}_n\Big)(\omega) \\ &= \Sigma_\omega \Big(1 + \frac{X(\omega)}{2\|X\|_\infty}\Big) P^*(\omega)\Big(\Sigma_1^N H_n.\Delta\tilde{S}_n\Big)(\omega). \end{aligned}$$

The '1' term on the right gives $E^*(\Sigma_1^N H_n.\Delta\tilde{S}_n)$, which is zero since this is a martingale transform of the E^*-martingale $\tilde{S}_n$. The 'X' term gives a multiple of the inner product

$$(X, \Sigma_1^N H_n.\Delta\tilde{S}_n),$$

which is zero as X is orthogonal to $\tilde{\mathcal{V}}$ and $\Sigma_1^N H_n.\Delta\tilde{S}_n \in \tilde{\mathcal{V}}$. By the Martingale Transform Lemma, $\tilde{S}_n$ is a P^{**}-martingale since H (previsible) is arbitrary. Thus P^{**} is a second equivalent martingale measure, different from P^*. So incompleteness implies non-uniqueness of equivalent martingale measures. //

Martingale Representation. To say that every contingent claim can be replicated means that every P^*-martingale (where P^* is the risk-neutral measure, which is unique) can be written, or *represented*, as a martingale transform (of the discounted prices) by the replicating (perfect-hedge) trading strategy H. In stochastic-process language, this says that all P^*-martingales can be *represented* as martingale transforms of discounted prices. Such Martingale Representation Theorems hold much more generally, and are very important. For the Brownian case, see VI and [RY], Ch. V.

Note. In the example of Chapter I, we saw that the simple option there could be replicated. More generally, in our market set-up, *all* options can be replicated – our market is *complete*. Similarly for the Black-Scholes theory below.

§4. The Fundamental Theorem of Asset Pricing; Risk-neutral valuation.

We summarise what we have learned so far. We call a measure P^* under which discounted prices $\tilde{S}_n$ are P^*-martingales a *martingale measure*. Such a P^* equivalent to the true probability measure P is called an *equivalent martingale measure*. Then
1 (**No-Arbitrage Theorem**: §2). If the market is *viable* (arbitrage-free), equivalent martingale measures P^* *exist*.
2 (**Completeness Theorem**: §3). If the market is *complete* (all contingent claims can be replicated), equivalent martingale measures are *unique*. Combining:

Theorem (Fundamental Theorem of Asset Pricing, FTAP). In a complete viable market, there exists a unique equivalent martingale measure P^* (or Q).

Let h (≥ 0, $\mathcal{F}_N$-measurable) be any contingent claim, H an admissible strategy replicating it:

$$V_N(H) = h.$$

As $\tilde{V}_n$ is the martingale transform of the P^*-martingale $\tilde{S}_n$ (by H_n), $\tilde{V}_n$ is a P^*-martingale. So $V_0(H)(= \tilde{V}_0(H)) = E^*(\tilde{V}_N(H))$. Writing this out in full:

$$V_0(H) = E^*(h/S^0_N).$$

More generally, the same argument gives $\tilde{V}_n(H) = V_n(H)/S^0_n = E^*[(h/S^0_N)|\mathcal{F}_n]$:

$$V_n(H) = S^0_n E^*(\frac{h}{S^0_N}|\mathcal{F}_n) \qquad (n = 0, 1, \cdots, N).$$

It is natural to call $V_0(H)$ above the *value* of the contingent claim h at time 0, and $V_n(H)$ above the value of h at time n. For, if an investor *sells* the claim h at time n for $V_n(H)$, he can follow strategy H to replicate h at time N and clear the claim. To sell the claim for *any other amount* would provide an arbitrage opportunity (as with the argument for put-call parity). So this value $V_n(H)$ is the *arbitrage price* (or more exactly, *arbitrage-free price* or *no-arbitrage price*); an investor selling for this value is *perfectly hedged*.