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Lecture 14 11.10.2014

§8. Doob Decomposition.

Theorem. Let $X = (X_n)$ be an adapted process with each $X_n \in L_1$. Then X has an (essentially unique) Doob decomposition

$$X = X_0 + M + A : \qquad X_n = X_0 + M_n + A_n \qquad \forall n \tag{D}$$

with M a martingale null at zero, A a previsible process null at zero. If also X is a submartingale ('increasing on average'), A is increasing: $A_n \leq A_{n+1}$ for all n, a.s.

Proof. If X has a Doob decomposition (D),

$$E[X_n - X_{n-1}|\mathcal{F}_{n-1}] = E[M_n - M_{n-1}|\mathcal{F}_{n-1}] + E[A_n - A_{n-1}|\mathcal{F}_{n-1}].$$

The first term on the right is zero, as M is a martingale. The second is $A_n - A_{n-1}$, since A_n (and A_{n-1}) is $\mathcal{F}_{n-1}$-measurable by previsibility. So

$$E[X_n - X_{n-1}|\mathcal{F}_{n-1}] = A_n - A_{n-1}, \tag{1}$$

and summation gives

$$A_n = \sum_1^n E[X_k - X_{k-1}|\mathcal{F}_{k-1}], \qquad a.s.$$

We use this formula to *define* (A_n), clearly previsible. We then use (D) to *define* (M_n), then a martingale, giving the Doob decomposition (D).

If X is a submartingale, the LHS of (1) is ≥ 0, so the RHS of (1) is ≥ 0, i.e. (A_n) is increasing. //

Note. 1. Although the Doob decomposition is a simple result in discrete time, the analogue in continuous time is deep (see Ch. V). This illustrates the contrasts that may arise between the theories of stochastic processes in discrete and continuous time.

2. Previsible processes are 'easy' (trading is easy if you can foresee price movements!). So the Doob Decomposition splits any (adapted) process X into two bits, the 'easy' (previsible) bit A and the 'hard' (martingale) bit M. Moral: martingales are everywhere!

3. Submartingales model favourable games, so are *increasing on average*. It 'ought' to be possible to split such a process into an *increasing* bit, and a remaining 'trendless' bit. It is – the trendless bit is the martingale.
4. This situation resembles that in Statistics, specifically Regression (see e.g. [BF]), where one has a decomposition

Data = Signal + noise = fitted value + residual.

§9. Examples.

1. *Simple random walk.*

Recall the simple random walk: $S_n := \sum_1^n X_k$, where the X_n are independent tosses of a fair coin, taking values ± 1 with equal probability 1/2. Suppose we decide to bet until our net gain is first $+1$, then quit. Let T be the time we quit; T is a stopping time.

The stopping-time T has been analysed in detail; see e.g.
[GS] GRIMMETT, G. R. & STIRZAKER, D.: *Probability and random processes*, OUP, 3rd ed., 2001 [2nd ed. 1992, 1st ed. 1982], §5.2.
From this, note:
(i) $T < \infty$ a.s.: the gambler will certainly achieve a net gain of $+1$ eventually;
(ii) $ET = +\infty$: the mean waiting-time till this happens is infinity. So:
(iii) No bound can be imposed on the gambler's maximum net loss before his net gain first becomes $+1$.

At first sight, this looks like a foolproof way to make money out of nothing: just bet till you get ahead (which happens eventually, by (i)), then quit. However, as a gambling strategy, this is hopelessly impractical: because of (ii), you need unlimited time, and because of (iii), you need unlimited capital – neither of which is realistic.

Notice that the Optional Stopping Theorem fails here: we start at zero, so $S_0 = 0$, $ES_0 = 0$; but $S_T = 1$, so $ES_T = 1$. This shows two things:
(a) The Optional Stopping Theorem does indeed need conditions, as the conclusion may fail otherwise [none of the conditions (i) - (iii) in the OST are satisfied in the example above],
(b) Any practical gambling (or trading) strategy needs to have some integrability or boundedness restrictions to eliminate such theoretically possible but practically ridiculous cases.

2. *The doubling strategy.*

The strategy of doubling when losing – *the* martingale, according to the Oxford English Dictionary (§3) has similar properties – and would be suicidal in practice as a result.

Chapter IV. MATHEMATICAL FINANCE IN DISCRETE TIME.

We follow [BK], Ch. 4.

§1. The Model.

It suffices, to illustrate the ideas, to work with a *finite* probability space $(\Omega, \mathcal{F}, \mathcal{P})$, with a finite number $|\Omega|$ of points ω, each with positive probability: $P(\{\omega\}) > 0$. We will use a finite time-horizon N, which will correspond to the expiry date of the options.

As before, we use a filtration $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_N$: we may (and shall) take $\mathcal{F}_0 = \{\emptyset, \Omega\}$, the trivial σ-field, $\mathcal{F}_N = \mathcal{F} = \mathcal{P}(\Omega)$ (here $\mathcal{P}(\Omega)$ is the *power-set* of Ω, the class of all $2^{|\Omega|}$ subsets of Ω: we need every possible subset, as they all (apart from the empty set) carry positive probability.

The financial market contains $d+1$ financial assets: a riskless asset (bank account) labelled 0, and d risky assets (stocks, say) labelled 1 to d. The prices of the assets at time n are random variables, $S_n^0, S_n^1, \cdots, S_n^d$ say [note that we use superscripts here as labels, *not* powers, and suppress ω for brevity], non-negative and $\mathcal{F}_n$-measurable [at time n, we know the prices S_n^i].

We take $S_0^0 = 1$ (that is, we reckon in units of our initial bank holding). We assume for convenience a constant interest rate $r > 0$ in the bank, so 1 unit in the bank at time 0 grows to $(1+r)^n$ at time n. So $1/(1+r)^n$ is the *discount factor* at time n.

Definition. A *trading strategy* H is a vector stochastic process $H = (H_n)_{n=0}^N = ((H_n^0, H_n^1, \cdots, H_n^d))_{n=0}^N$ which is *predictable* (or *previsible*): each H_n^i is $\mathcal{F}_{n-1}$-measurable for $n \geq 1$.

Here H_n^i denotes the number of shares of asset i held in the portfolio at time n – to be determined on the basis of information available *before* time n; the vector $H_n = (H_n^0, H_n^1, \cdots, H_n^d)$ is the *portfolio* at time n. Writing $S_n = (S_n^0, S_n^1, \cdots, S_n^d)$ for the vector of prices at time n, the *value* of the portfolio at time n is the scalar product

$$V_n(H) = H_n.S_n := \Sigma_{i=0}^d H_n^i S_n^i.$$

The *discounted value* is

$$\tilde{V}_n(H) = \beta_n(H_n.S_n) = H_n.\tilde{S}_n,$$

where $\beta_n := 1/S_n^0$ and $\tilde{S}_n = (1, \beta_n S_n^1, \cdots, \beta_n S_n^d)$ is the vector of discounted prices.
Note. The *previsibility* of H reflects that there is no *insider trading*.

Definition. The strategy H is *self-financing*, $H \in SF$, if

$$H_n.S_n = H_{n+1}.S_n \qquad (n = 0, 1, \cdots, N-1).$$

Interpretation. When new prices S_n are quoted at time n, the investor adjusts his portfolio from H_n to H_{n+1}, without bringing in or consuming any wealth.
Note.

$$\begin{aligned} V_{n+1}(H) - V_n(H) &= H_{n+1}.S_{n+1} - H_n.S_n \\ &= H_{n+1}.(S_{n+1} - S_n) + (H_{n+1}.S_n - H_n.S_n). \end{aligned}$$

For a self-financing strategy, the second term on the right is zero. Then the LHS, the net increase in the value of the portfolio, is shown as due only to the price changes $S_{n+1} - S_n$. So for $H \in SF$,

$$V_n(H) - V_{n-1}(H) = H_n(S_n - S_{n-1}),$$

$$\Delta V_n(H) = H_n.\Delta S_n, \qquad V_n(H) = V_0(H) + \Sigma_1^n H_j.\Delta S_j$$

and $V_n(H)$ is the *martingale transform* of S by H (III.6). Similarly with discounting:

$$\Delta \tilde{V}_n(H) = H_n.\Delta \tilde{S}_n, \qquad \tilde{V}_n(H) = V_0(H) + \Sigma_1^n H_j.\Delta \tilde{S}_j$$

$(\Delta \tilde{S}_n := \tilde{S}_n - \tilde{S}_{n-1} = \beta_n S_n - \beta_{n-1} S_{n-1})$.

As in I, we are allowed to borrow (so S_n^0 may be negative) and sell short (so S_n^i may be negative for $i = 1, \cdots, d$). So it is hardly surprising that if we decide what to do about the risky assets, the bank account will take care of itself, in the following sense.

Proposition. If $((H_n^1, \cdots, H_n^d))_{n=0}^N$ is predictable and V_0 is $\mathcal{F}_0$-measurable, there is a unique predictable process $(H_n^0)_{n=0}^N$ such that $H = (H^0, H^1, \cdots, H^d)$ is self-financing with initial value V_0.