Lecture 32. 25.3.2015.

Proof of PNT with remainder (concluded).

As the residue at $s = 1$ is $x$ (as $-\zeta'/\zeta$ has a simple pole at 1 of residue 1), Cauchy’s Residue Theorem gives

$$\frac{1}{2\pi i} \int_{k-iT}^{k+iT} -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds = x - \frac{1}{2\pi i} \int_{P} -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds,$$

with $P$ the polygonal path with vertices $k \pm iT, 1 - c_0/\log T \pm iT$. On $P$,

$$-\zeta'(s)/\zeta(s) << \log T \quad \text{(IV.4 L31)}.$$

The horizontal parts have length

$$k - (1 - c_0/\log T) = \frac{1}{\log x} + \frac{c_0}{\log T},$$

and on them $x^s/s$ is of order $x^\sigma/T$ ($s = \sigma + i\tau$ with $\sigma$ bounded and $\tau = \pm T$; $|x^s| = x^\sigma << x^k = e^{k\log x} = e x << x$, as $k = 1 + 1/\log x$). So the horizontal parts contribute

$$<< \left( \frac{1}{\log x} + \frac{1}{\log T} \right) \frac{x}{T} \log T = \frac{x}{T} \left( 1 + \frac{\log T}{\log x} \right) << \frac{x \log T}{T}.$$

For the vertical part, we split into $|t| \leq 1$ and $1 \leq |t| \leq T$. For the first, as $T, \log T$ are large (so the line is only just to the left of the 1-line), we are near the simple pole of $-\zeta'/\zeta$ at 1 (of residue 1), so $-\zeta'(s)/\zeta(s) = O(1/(s-1))$. So this integral is

$$I_1 << x \int_{-1}^{1} \frac{dt}{|it - c_0/\log T|} << x \log T.$$

For the second, we use $-\zeta'/\zeta << \log T$; the integral is, writing $D := \{ s : \sigma = a := 1 - c_0/\log T, -T \leq \tau \leq -1 \text{ or } 1 \leq \tau \leq T,$

$$I_2 = \int_{D} -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds << (\log T).x^a \int_{-T}^{T} \frac{dt}{1 + |t|} << x^a(\log T)^2.$$

So the vertical part is $<< x^a(\log T)^2$.

Choice of $T$. 

1
Throughout, $x$ is as in PNT, and will $\to \infty$; we will let $T \to \infty$ also, but can choose how fast. We choose
\[ T := \exp\{\sqrt{c_0 \log x}\} : \quad \log T = \sqrt{c_0 \log x} \ll \sqrt{\log x}; \]
\[ \frac{\log T}{T} \ll \sqrt{\log x}. \exp\{\sqrt{c_0 \log x}\} \ll \exp\{\sqrt{c_1 \log x}\} \quad (\text{for all } c_1 < c_0). \]

The second error term in ‘Perron for $-\zeta'/\zeta$’ is now the dominant one. As $x$ grows more slowly than any $\exp\{x^a\}$ for $a > 0$, $\log x$ grows more slowly than any $\exp\{(\log x)^a\}$. We can thus absorb the $x \log x$ and the $(\log T)/T$ above into
\[ x \exp\{-\sqrt{c \log x}\} \quad (c < c_1 < c_0), \]
that is, for any $c < c_0$.

**Horizontal contributions:**
\[ \ll \frac{x \log T}{T} \leq \frac{x \sqrt{\log x}}{\exp\{\sqrt{c_0 \log x}\}} \ll x \exp\{-\sqrt{c \log x}\}. \]

**Vertical contributions:**
\[ \ll x^a (\log T)^2. \]

Now
\[ x^a = x \cdot x^{-c_0/\log T} = x \cdot \exp\{-\frac{c_0}{\log T} \cdot \log x\} = x \cdot \exp\{-\frac{c_0}{\sqrt{c_0 \log x}} \cdot \log x\} \ll x \exp\{-\sqrt{c_0 \log x}\} : \]
\[ x^a (\log T)^2 = x \log x \cdot \exp\{-\sqrt{c_0 \log x}\} \ll x \exp\{-\sqrt{c \log x}\} \]
for any $c < c_0$, as above. Combining,
\[ \psi(x) - x \ll x \exp\{-\sqrt{c \log x}\}. \]

**Note.** 1. As a very special case, the result above includes
\[ \pi(x) = \text{li}(x) + O(x/\log^2 x) = x/\log x + O(x/\log^2 x). \]

There seems to be no quicker way to obtain this crude-looking form of PNT with remainder than by specialisation of the classical result proved above.

2. We used the classical ZFR (Hadamard-de la Vallée-Poussin, 1896) here. We mentioned above (IV.3 L30) the best ZFR known (Vinogradov, Korobov, 1958), proved by Complex Analysis as here. The best error term obtained so far by elementary methods (not using Complex Analysis – see III.1) gives $O(x \exp\{-c \log^a x\})$ with $\alpha = 1/6 - \epsilon$ (Lavrik and Sobirov, 1973). By Turán’s method (IV.3 L30), this still yields a non-trivial zero-free region (though not, of course, as good as the classical one or the best-known one).