Lecture 3. 15.1.2015

Abel’s Summation Formula. If $f$ has a continuous derivative on $[y, x],
\[ \sum_{y < r \leq x} a_r f_r = A(x) f(x) - A(y) f(y) - \int_y^x A(t) f'(t) dt. \]

Proof. Let $m = \lfloor y \rfloor$, $x = \lceil n \rceil$, with $[\cdot]$ the integer part. Then
$\sum_{y < r \leq n} a_r f_r = \sum_{r=m+1}^{n} a_r f_r$. As $A(x) := \sum_{r \leq x} a_r$, $A(t) = A(r)$ for $r \leq t < r + 1$. So
$\sum_{m+1}^{n} A_r (f_r - f_{r+1}) = \sum_{m+1}^{n-1} A(r) \int_r^{r+1} f'(t) dt$
$= \sum_{m+1}^{n-1} \int_r^{r+1} A(t) f'(t) dt$ as $A$ is constant on $(r, r + 1)$
$= - \int_{m+1}^{n} A(t) f'(t) dt.$

Similarly, for $n \leq t \leq x$ $A(t) = A(n)$, so
$A(x) f(x) - A(n) f(n) = A(n) [f(x) - f(n)] = \int_n^x A(t) f'(t) dt,$
and for $m \leq t \leq y$ $A(t) = A(m)$, so
$A(m) f(m + 1) - A(y) f(y) = A(m) [f(m + 1) - f(y)] = \int_y^{m+1} A(t) f'(t) dt.$

Now substitute into (*) in the proof of Abel’s Lemma for $A_n f_n - A_m f_{m+1}$. //

Stieltjes integrals. If $\alpha$ is non-decreasing and right-continuous, and we replace
$x_{i+1} - x_i$ in the Riemann integral everywhere by $\alpha((x_i, x_{i+1}]) := \alpha(x_{i+1}) - \alpha(x_i)$, we obtain the Riemann-Stieltjes (RS) integral (there is a Lebesgue-Stieltjes (LS) version). The integration-by-parts formula
$\int_{[a,b]} f dg = f(b) g(b) - f(a) g(a) - \int_{[a,b]} g df$
holds for Stieltjes integrals (see e.g. [Ten], p.1, 107). When $\alpha$ is a step-function $\alpha(x) = \sum_{n \leq x} a_n$ and $f(x) = \int_x^z f'(u) du$ is absolutely continuous, we recover the result above.
§4. The Integral Test and Euler’s Constant

The Integral Test: If $f > 0$ and is monotonic decreasing on $[1, \infty)$, then:
(i) $\int_1^\infty f(x)\,dx$ and $\sum_1^\infty f(n)$ converge or diverge together;
(ii) $\sum_1^n f(r) - \int_1^n f(x)\,dx \to l \in [0, f(1)]$ as $n \to \infty$.

Proof. (i) As $f$ is monotonic, it is integrable on each $[1, x]$. If $n - 1 \leq x \leq n$,
$$f(n - 1) \geq f(x) \geq f(n): \quad f(n - 1) \geq \int_{n-1}^n f(x)\,dx \geq f(n).$$

Sum from 1 to $n - 1$:
$$\sum_{r=1}^{n-1} f(r) \geq \int_1^n f \geq \sum_{r=2}^n f(r): \quad \sum_{r=1}^n f(r) - f(n) \geq \int_1^n f \geq \sum_{r=1}^n f(r) - f(1). \quad (*)$$

If $\sum_1^\infty f(r) < \infty$, the LH inequality gives $\int_1^\infty f(x)\,dx < \infty$.
If $\int_1^\infty f(x)\,dx < \infty$, the RH inequality gives $\sum_1^\infty f(r) < \infty$. For (ii),
$$f(1) \geq \phi(n) := \sum_{r=1}^n f(r) - \int_1^n f \geq f(n) \geq 0.$$

Then by ($\ast$),
$$\phi(n) - \phi(n - 1) = f(n) - \int_{n-1}^n f(x)\,dx \leq 0, \quad 0 \leq \phi(n) \leq f(1),$$
So $\phi(n)$ is bounded and decreasing, so it is convergent: $\phi(n) \downarrow l \in [0, f(1)]$. //

Taking $f(x) \equiv 1/x$, the limit is defined as Euler’s constant, $\gamma$. Then [J]:

Corollary (Euler’s Constant).
$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \log n \to \gamma \quad (n \to \infty).$$

$$0 < \sum_1^N \frac{1}{n} - \log N < 1; \quad \sum_1^N \frac{1}{n} = \log N + \gamma + \frac{1}{2N} + O\left(\frac{1}{2N}\right).$$