Theorem (Maximum Modulus Theorem: Local form). If \( f \) is holomorphic in \( N(a, R) \), and
\[
|f(z)| \leq |f(a)| \quad \forall z \in N(a, R)
\]
– then \( f \) is constant.

Proof. Fix \( r, 0 < r < R \). By CIF,
\[
f(a) = \frac{1}{2\pi i} \int_{\gamma(a,r)} \frac{f(z) \, dz}{z-a}
\]
\[
= \frac{1}{2\pi i} \int_{0}^{2\pi} f(a + re^{i\theta}) \, ire^{i\theta} \, d\theta
\]
\[
= \frac{1}{2\pi i} \int_{0}^{2\pi} f(a + re^{i\theta}) \, d\theta.
\]
So
\[
|f(a)| \leq \frac{1}{2\pi} \left| \int_{0}^{2\pi} \right| \leq \frac{1}{2\pi} \int_{0}^{2\pi} |f(a + re^{i\theta})| \, d\theta \leq \frac{1}{2\pi} \int_{0}^{2\pi} |f(a)| \, d\theta = |f(a)|,
\]
by hypothesis. So both inequalities are equalities:
\[
\int_{0}^{2\pi} (|f(a)| - |f(a + re^{i\theta})|) \, d\theta = 0.
\]
The integrand is continuous (as \( f \) is holomorphic), and \( \geq 0 \) (by hypothesis). So it is \( \equiv 0 \). So
\[
|f(a)| - |f(a + re^{i\theta})| \quad \forall \theta \in [0, 2\pi], \ r \in (0, R).
\]
So \( |f| \) is constant.

If \( f = u + iv, \ |f|^2 = u^2 + v^2 \) is constant, \( c^2 \) say. So (applying \( \partial/\partial x \) and \( \partial/\partial y \)):
\[
2uu_x + 2vv_x = 0, \quad 2uu_y + 2vv_y = 0.
\]
Using the Cauchy-Riemann equations,
\[
uu_x - vv_y = 0, \quad uu_y + vu_x = 0.
\]
Multiply the first by \( u \), the second by \( v \) and add:
\[
(u^2 + v^2)u_x = 0, \quad c^2u_x = 0.
\]
If \( c = 0, \ f = 0, \) constant. If \( c \neq 0, \ u_x = 0. \) Similarly, \( u_y = 0. \) So \( u \) is constant. Similarly, \( v \) is constant. So \( f = u + iv \) is constant. //
Theorem (Maximum Modulus Theorem). If $D$ is a bounded domain and $f$ is holomorphic on $D$ and continuous on its closure $\overline{D}$ – then $|f|$ attains its maximum on the boundary $\partial D := \overline{D} \setminus D$.

Proof. $D$ is bounded, so $\overline{D}$ is closed and bounded, so is compact (Heine-Borel Thm.). As $|f|$ is continuous on the compact set $\overline{D}$, it attains its supremum $M$ on $\overline{D}$, at $a$ say.

Assume $a \notin \partial D$ (which will give a contradiction). Then $a \in D$, open, so $N(a, R) \subset D$ for some $R > 0$. So $|f|$ attains its maximum on $N(a, R)$ at $a$. By the above Local Form, $f$ is constant on $N(a, R)$. So by the Identity Theorem, $f \equiv \text{constant}$.

If $f$ is non-constant, this gives the required contradiction, showing $|f|$ attains its maximum on the boundary $\partial D$. If $f$ is constant, all points are maxima, a trivial case. //

Theorem (Minimum Modulus Theorem). If $f$ is holomorphic and non-constant on a bounded domain $D$, then $|f|$ attains its minimum either at a zero of $f$ or on the boundary.

Proof. If $f$ has a zero in $D$, $|f|$ attains its minimum there. If not, apply the Maximum Modulus Theorem to $1/f$.

Theorem (Maximum Modulus Theorem for Harmonic Functions). If $D$ is a bounded domain, $u$ is harmonic in $D$ and continuous on $\overline{D}$, and $u \leq M$ on $\partial D$: then $u \leq M$ on $\overline{D}$. That is, $u$ attains its maximum on the boundary $\partial D$.

Proof: similar to the above – omitted.

Applications.

Applications include asymptotics, in particular the Saddlepoint method (Riemann, posthumous, 1892) and Method of steepest descents (P. DEBYE, 1909). Suppose we have to estimate a line integral, of a holomorphic function $f$ along a curve $\gamma$ in a bounded region of holomorphy $D$. We look for a stationary point $z_0$ of the integrand $f = u + iv$ on $\gamma$. As points on $\gamma$ (closed) are interior points of $D$ (open), $u$ attains its maximum on the boundary. So $z_0 \in \gamma$ is not a maximum. Arguing similarly for $-f$, it is not a minimum, so must be a saddle-point (see Calculus of Several Variables in Real Analysis). The level curves (contours) $u$ constant near $z_0$ cut the level curves $v$ constant orthogonally, and these are paths of steepest descent (as with contours on an OS map). As in the Deformation Lemma, we may deform $\gamma$ to such a path of steepest descent. We must refer for further detail to a book or course on Asymptotics. Suffice it to point out here that applications include Stirling’s formula for the factorial, or the Gamma function:

$$n! \sim \sqrt{2\pi}e^{-n}n^{n+\frac{1}{2}} \quad (n \to \infty), \quad \Gamma(x) \sim \sqrt{2\pi}e^{-x}x^{x-\frac{1}{2}} \quad (x \to \infty)$$

(James STIRLING (1692-1770) in 1730).