1-1 MULTIVARIATE TRANSFORMATIONS

Given a collection of variables \((X_1, \ldots, X_k)\) with range \(X^{(k)}\) and joint pdf \(f_{X_1, \ldots, X_k}\) we can construct the pdf of a transformed set of variables \((Y_1, \ldots, Y_k)\) using the following steps:

1. Write down the set of transformation functions \(g_1, \ldots, g_k\)
   \[
   Y_1 = g_1 (X_1, \ldots, X_k) \\
   \vdots \\
   Y_k = g_k (X_1, \ldots, X_k)
   \]

2. Write down the set of inverse transformation functions \(g_1^{-1}, \ldots, g_k^{-1}\)
   \[
   X_1 = g_1^{-1} (Y_1, \ldots, Y_k) \\
   \vdots \\
   X_k = g_k^{-1} (Y_1, \ldots, Y_k)
   \]

3. Consider the joint range of the new variables, \(Y^{(k)}\).

4. Compute the Jacobian of the transformation: first form the matrix of partial derivatives
   \[
   D_y = \begin{bmatrix}
   \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_k} \\
   \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_2}{\partial y_k} \\
   \vdots & \vdots & \ddots & \vdots \\
   \frac{\partial x_k}{\partial y_1} & \frac{\partial x_k}{\partial y_2} & \cdots & \frac{\partial x_k}{\partial y_k}
   \end{bmatrix}
   \]
   where, for each \((i, j)\)
   \[
   \frac{\partial x_i}{\partial y_j} = \frac{\partial}{\partial y_j} \{g_i^{-1} (y_1, \ldots, y_k)\}
   \]
   and then set \(|J (y_1, \ldots, y_k)| = |\det D_y|\)

   Note that
   \[
   |\det D_y| = |\det D_y^T|
   \]
   so that an alternative but equivalent Jacobian calculation can be carried out by forming \(D_y^T\). Note also that
   \[
   |J (y_1, \ldots, y_k)| = \frac{1}{|J (x_1, \ldots, x_k)|}
   \]
   where \(J (x_1, \ldots, x_k)\) is the Jacobian of the transformation regarded in the reverse direction (that is, if we start with \((Y_1, \ldots, Y_k)\) and transform to \((X_1, \ldots, X_k)\)).

5. Write down the joint pdf of \((Y_1, \ldots, Y_k)\) as
   \[
   f_{Y_1, \ldots, Y_k} (y_1, \ldots, y_k) = f_{X_1, \ldots, X_k} (g_1^{-1} (y_1, \ldots, y_k), \ldots, g_k^{-1} (y_1, \ldots, y_k)) \times |J (y_1, \ldots, y_k)|
   \]
   for \((y_1, \ldots, y_k) \in Y^{(k)}\).
EXAMPLE 1 Suppose that \( X_1 \) and \( X_2 \) have joint pdf

\[
f_{X_1, X_2}(x_1, x_2) = 2, \quad 0 < x_1 < x_2 < 1,
\]

and zero otherwise. Compute the joint pdf of random variables

\[
Y_1 = \frac{X_1}{X_2}, \quad Y_2 = X_2.
\]

SOLUTION

1. Given that \( X^{(2)} \equiv \{(x_1, x_2) : 0 < x_1 < x_2 < 1\} \) and

\[
g_1(t_1, t_2) = \frac{t_1}{t_2}, \quad g_2(t_1, t_2) = t_2.
\]

2. Inverse transformations:

\[
Y_1 = X_1/X_2, \quad Y_2 = X_2 \quad \Leftrightarrow \quad \begin{cases} X_1 = Y_1 Y_2 \\ X_2 = Y_2 \end{cases}
\]

and thus

\[
g_1^{-1}(t_1, t_2) = t_1 t_2, \quad g_2^{-1}(t_1, t_2) = t_2.
\]

3. Range: to find \( Y^{(2)} \) consider point by point transformation from \( X^{(2)} \) to \( Y^{(2)} \). For a pair of points \((x_1, x_2) \in X^{(2)}\) and \((y_1, y_2) \in Y^{(2)}\) linked via the transformation, we have

\[
0 < x_1 < x_2 < 1 \Leftrightarrow 0 < y_1 y_2 < y_2 < 1
\]

and hence we can extract the inequalities

\[
0 < y_2 < 1 \text{ and } 0 < y_1 < 1 \quad \therefore \quad Y^{(2)} \equiv (0, 1) \times (0, 1).
\]

4. The Jacobian for points \((y_1, y_2) \in Y^{(2)}\) is found from

\[
D_y = \begin{bmatrix}
\frac{\partial x_1}{\partial y_1} & \frac{\partial x_2}{\partial y_1} \\
\frac{\partial x_1}{\partial y_2} & \frac{\partial x_2}{\partial y_2}
\end{bmatrix} = \begin{bmatrix} y_2 & y_1 \\ 0 & 1 \end{bmatrix} \Rightarrow |J(y_1, y_2)| = |\text{det } D_y| = |y_2| = y_2.
\]

Note that for points \((x_1, x_2) \in X^{(2)}\) is

\[
D_x = \begin{bmatrix}
\frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\
\frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2}
\end{bmatrix} = \begin{bmatrix} 1 & x_1 \\ x_2 & x_2 \end{bmatrix} \Rightarrow |J(x_1, x_2)| = |\text{det, } D_x| = \frac{1}{x_2} = \frac{1}{x_2},
\]

verifying that

\[
|J(y_1, y_2)| = \frac{1}{|J(x_1, x_2)|}.
\]

5. Finally, we have

\[
f_{Y_1, Y_2}(y_1, y_2) = f_{X_1, X_2}(y_1 y_2, y_2) \times y_2 = 2y_2, \quad 0 < y_1 < 1, 0 < y_2 < 1,
\]

and zero otherwise.
EXAMPLE 2 Suppose that $X_1$ and $X_2$ are independent and identically distributed random variables defined on $\mathbb{R}^+$ each with pdf of the form

$$f_X(x) = \sqrt{\frac{1}{2\pi x}} \exp\left\{ -\frac{x}{2} \right\}, \quad x > 0,$$

and zero otherwise. Compute the joint pdf of random variables $Y_1 = X_1$ and $Y_2 = X_1 + X_2$.

SOLUTION

1. Given that $X(2) \equiv \{(x_1, x_2) : 0 < x_1, 0 < x_2\}$ and

$$g_1(t_1, t_2) = t_1, \quad g_2(t_1, t_2) = t_1 + t_2.$$

2. Inverse transformations:

$$Y_1 = X_1 \quad \Rightarrow \quad \{ X_1 = Y_1 \} \quad \{ X_2 = Y_2 - Y_1 \}$$

3. Range: to find $Y(2)$ consider point by point transformation from $X(2)$ to $Y(2)$. For a pair of points $(x_1, x_2) \in X(2)$ and $(y_1, y_2) \in Y(2)$ linked via the transformation, as both original variables are strictly positive, we can extract the inequalities

$$0 < y_1 < y_2 < \infty.$$

4. The Jacobian for points $(y_1, y_2) \in Y(2)$ is found from

$$D_y = \begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \Rightarrow |J(y_1, y_2)| = |\det D_y| = |1| = 1.$$

5. Finally, we have for $0 < y_1 < y_2 < \infty$,

$$f_{Y_1, Y_2}(y_1, y_2) = f_{X_1, X_2}(y_1 - y_1, y_2 - y_1) \times 1 = f_{X_1}(y_1) \times f_{X_2}(y_2 - y_1) \quad \text{by independence}$$

$$= \sqrt{\frac{1}{2\pi y_1}} \exp\left\{ -\frac{y_1}{2} \right\} \sqrt{\frac{1}{2\pi (y_2 - y_1)}} \exp\left\{ -\frac{(y_2 - y_1)}{2} \right\}$$

$$= \frac{1}{2\pi \sqrt{y_1(y_2 - y_1)}} \exp\left\{ -\frac{y_2}{2} \right\},$$

and zero otherwise.
Here, for $y_2 > 0$,

$$f_{Y_2}(y_2) = \int f_{Y_1,Y_2}(y_1,y_2) \, dy_1 = \int_0^{y_2} \frac{1}{2\pi} \frac{1}{\sqrt{y_1 (y_2 - y_1)}} \exp\left\{-\frac{y_2}{2}\right\} \, dy_1$$

$$= \frac{1}{2\pi} \exp\left\{-\frac{y_2}{2}\right\} \int_0^{y_2} \frac{1}{\sqrt{y_1 (y_2 - y_1)}} \, dy_1$$

$$= \frac{1}{2\pi} \exp\left\{-\frac{y_2}{2}\right\} \int_0^1 \frac{1}{\sqrt{t y_2 (y_2 - ty_2)}} y_2 \, dt \quad \text{setting } y_1 = ty_2$$

$$= \frac{1}{2\pi} \exp\left\{-\frac{y_2}{2}\right\} \int_0^1 \frac{1}{\sqrt{t (1 - t)}} \, dt$$

$$= \frac{1}{2} \exp\left\{-\frac{y_2}{2}\right\},$$

as (by MAPLE, or further transformations)

$$\int_0^1 \frac{1}{\sqrt{t (1 - t)}} \, dt = \pi.$$