K-STABILITY AND ITS RELATIONS WITH CONSTANT SCALAR CURVATURE KÄHLER ORBIFOLDS

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Academic Year 2014/2015

The research leading to these results has received funding from the European Research Council under the European Union’s Seventh Framework Programme (FP7/2007-2013)/ERC Grant Agreement no. 307119
To my parents,
for their love and support
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Introduction

This thesis is devoted to the study of some aspects of one of the major problems in Kähler Geometry of the recent years: providing necessary and sufficient conditions for the solvability of the constant scalar curvature equation on Kähler manifolds and orbifolds. Since the work of E. Calabi in the 1970’s an enormous quantity of results have been obtained and the question whether one can find algebro-geometric characterisations for the existence of canonical metrics on complex manifolds and orbifolds has attracted the attention of many researchers, indeed his problem perfectly fits in the frameworks of both Algebraic and Differential Geometry. Many applications of these results can be found in the context of Theoretical and Mathematical Physics.

The present work is mainly inspired by the papers of Ross and Thomas [20] and Stoppa [26], which can both be considered as part of the mathematical literature devoted to the study of a conjecture formulated by S. T. Yau, S. Donaldson and G. Tian, the so called YDT conjecture. This conjecture aims at relating the existence of constant scalar curvature Kähler (cscK) metrics on a polarised manifold to an algebro-geometric notion of stability (“K-polystability”). One of the most important results concerning this topic is Donaldson’s Theorem, which states that if a polarised manifold is cscK, then it is K-semistable, [11]. The result in [26] provides a partial extension of Donaldson’s result: indeed it is proved that if a polarised manifold is cscK then it is K-stable, provided that the automorphism group is discrete. On the other hand, Ross and Thomas, in [20] were able to prove that the problem concerning the solvability of the cscK equation still makes sense in the setting of cyclic orbifolds. More importantly, they could extend Donaldson’s Theorem: thus, the cscK condition implies K-semistability also for polarised cyclic orbifolds. The relation between the two aforementioned papers is well synthesized by the following sentence, contained in the Introduction of [20]:

An improvement [of Donaldson’s result] by Stoppa says that, as long as one assumes a discrete automorphism group, the existence of a cscK metric actually implies K-stability. It is natural to ask if this too can be extended to orbifolds.

Indeed, the main purpose of this thesis is to address the problem of extending the results of [26] and [27] to the setting of orbifolds with cyclic stabilisers in codimension one. Such an extension is far from being immediately obtained; in fact, the proof of the K-stability Theorem is made of many steps: thus, each of them has to be analysed in order to understand if the argument is still valid in the context of orbifolds.
The arguments used in [27] constitute the so called "blow-up method" and our project was originally aimed at studying the possibility of completely extending it to the orbifold setting. On the other hand, one of the main tools used in the blow-up method is an analytical theorem by Arezzo and Pacard, [2]: finding a proof for the orbifold version of this result is a subtle problem, since it would involve the use of sophisticated techniques of analysis of PDEs, the study of which is beyond the purposes of this thesis. Given this, we focused our study on the blow-up formula, which is another fundamental step in the proof of [27]. In this framework, a formula for the behaviour of K-stability under blowing up is presented (Theorem 3.2). It is only a partial extension of the blow-up formula, since it holds true under certain smoothness hypotheses, but it still takes into account interesting geometric cases: indeed it can be applied to study orbifold slope stability and its relations with orbifold K-stability.

The present work is organised as follows:

- in the first chapter we present a self-contained introduction to the theory of orbifolds. More precisely, we focus our attention on complex orbifolds with cyclic stabilisers and prove an orbifold version of Kodaira Embedding Theorem, a fundamental tool in the definition of K-stability;

- in the second chapter we give the notion of test configuration and define K-(semi)stability both for manifolds and orbifolds. Furthermore, we recollect the main results of the theory, which inspired the study conducted in this thesis;

- finally, the third chapter contains the original part of our work: we give an alternative proof of the blow-up formula, carried forward with techniques that can be efficiently extended to the orbifold setting. We present our result under the name of "orbifold blow-up formula". 
Chapter 1

Orbifolds

In this first chapter we will deal with the basic theory of orbifolds, which constitute one of the central topics of this thesis. We do not aim at giving a complete treatment of this beautiful subfield of Geometry, which would occupy too many pages, but rather a self-contained exposition of the main and most important definitions and theorems, which are needed to understand the material treated in our dissertation. Our main references for this chapter will be [1], [4], [5], and [20].

Orbifolds lie at the intersection of many different areas of Mathematics, including Algebraic and Differential Geometry, Topology, Algebra and String Theory. The word orbifold was coined relatively recently: indeed, the notion of orbifold was introduced by Satake [22] in 1956 under the name of V-manifold. Subsequently Satake developed Riemannian geometry on V-manifolds and his theory culminated with a proof of an orbifold version of the Gauss-Bonnet Theorem. At the same time, complex V-manifolds were introduced and generalized versions of Hodge decomposition theorem and Kodaira’s projective embedding theorem were proved. In the late 1970’s, orbifolds were used by Thurston in his geometrization program for threefolds. Although orbifolds were already clearly important objects in Mathematics, interest in them was dramatically increased by their role in String Theory. Indeed, in 1985 Dixon, Harvey, Vafa and Witten built a conformal field theory model on singular spaces such as $\mathbb{T}^6/G$, the quotient of the six-dimensional torus by a smooth action of a finite group. This is just one of the numerous examples that can be provided to highlight how important and multidisciplinary the theory of orbifolds is.

1.1 Basic definitions

Although we will be mostly interested in complex orbifolds, when possible we will give definitions and state theorems that hold true in both cases of real and complex orbifolds. Given this, let $\mathbb{F}$ denote either the field $\mathbb{R}$ of real numbers or $\mathbb{C}$ of complex numbers.
Definition 1.1. Let $X$ be a paracompact Hausdorff space. An orbifold chart or local uniformizing system on $X$ is a triple $(\tilde{U}, \Gamma, \phi)$, where $\tilde{U}$ is a connected open subset of $\mathbb{F}^n$ containing the origin, $\Gamma$ is a finite group acting smoothly and effectively on $\tilde{U}$ and $\phi: \tilde{U} \to U$ is a continuous map onto an open set $U \subset X$ such that $\phi \circ \gamma = \phi$ for all $\gamma \in \Gamma$ and the induced natural map from $\tilde{U}/\Gamma$ onto $U$ is a homeomorphism. An injection or embedding between to such charts $(\tilde{U}, \Gamma, \phi)$ and $(\tilde{U}', \Gamma', \phi')$ is a smooth (or holomorphic) embedding $\lambda: \tilde{U} \to \tilde{U}'$ such that $\phi' \circ \lambda = \phi$.

Definition 1.2. An atlas on $X$ is a family $\mathcal{U} = \{U_i, \Gamma_i, \phi_i\}_{i \in I}$ such that

1. $X = \bigcup_{i \in I} \phi_i(U_i)$,
2. given two charts $(\tilde{U}_i, \Gamma_i, \phi_i)$ and $(\tilde{U}_j, \Gamma_j, \phi_j)$ with $U_i = \phi_i(\tilde{U}_i)$ and $U_j = \phi_j(\tilde{U}_j)$ and a point $x \in U_i \cap U_j$, there exists and open neighbourhood $U_k$ of $x$ and a chart $(\tilde{U}_k, \Gamma_k, \phi_k)$ such that there are injections $\lambda_{ik}: (\tilde{U}_k, \Gamma_k, \phi_k) \to (\tilde{U}_i, \Gamma_i, \phi_i)$ and $\lambda_{jk}: (\tilde{U}_k, \Gamma_k, \phi_k) \to (\tilde{U}_j, \Gamma_j, \phi_j)$.

Definition 1.3. An atlas $\mathcal{U}$ is said to be a refinement of an atlas $\mathcal{V}$ if there exists an injection of every chart of $\mathcal{U}$ into some chart of $\mathcal{V}$. Two atlases are said to be equivalent if they have a common refinement. A smooth orbifold (or V-manifold) is the datum of a paracompact Hausdorff space $X$ equipped with an equivalence class of equivalent atlases and we will use the notation $X = (X, \mathcal{U})$. If every finite group $\Gamma$ consists of orientation preserving diffeomorphisms and there is an atlas such that all injections are orientation preserving then the orbifold is said to be orientable.

Remark 1.1. Since smooth actions are locally smooth, any orbifold has an atlas consisting of linear charts, by which we mean charts of the form $(\mathbb{F}^n, \Gamma, \phi)$, where $\Gamma$ acts on $\mathbb{F}^n$ via orthogonal representation, i.e. $\Gamma \subset O(n)$ if $\mathbb{F} = \mathbb{R}$ or $\Gamma \subset U(n)$ if $\mathbb{F} = \mathbb{C}$.

Remark 1.2. Notice that, once fixed an orbifold chart $(\tilde{U}, \Gamma, \phi)$, any $\gamma \in \Gamma$ satisfies $\phi(\gamma \cdot x) = \phi(x)$ for any $x \in \tilde{U}$. Thus, $\gamma$ defines an injection $x \mapsto \gamma \cdot x$, which will be denoted by $\lambda_\gamma$.

The following is an important lemma for the study of orbifolds, but, since it is rather technical result, we will omit its proof, for which we refer to [16].

Lemma 1.1. Let $\lambda_1, \lambda_2: (\tilde{U}, \Gamma, \phi) \to (\tilde{U}', \Gamma', \phi')$ be two injections. Then there exists a unique $\gamma' \in \Gamma'$ such that $\lambda_2 = \gamma \circ \lambda_1$.

The previous result enables us to define the isotropy group: let $(X, \mathcal{U})$ be an orbifold and $x \in X$ a point. Let $(\phi(\tilde{U}) = U, \Gamma, \phi)$ be an orbifold chart around $x$ and choose $p \in \phi^{-1}(x)$; then, up to conjugacy, the isotropy group $\Gamma_p = \{ \gamma \in \Gamma, \gamma \cdot p = p \} \subset \Gamma$ depends only on $x$ and accordingly we will denote this isotropy subgroup by $\Gamma_x$. Even though $\Gamma_x$ is only defined up to conjugacy, $|\Gamma_x|$ is well defined and is called order at $x$ and denoted $\text{ord}_x(X)$. Thus, we have the following.

Definition 1.4. A point of $X$ is said to be a singular point if its isotropy subgroup is non-trivial. Those points for which $\Gamma_x = \{ e \}$ are called regular points. The set of singular points is called orbifold singular locus and is denoted by $\Sigma^{orb}(X)$. 

Remark 1.3. As one might expect, if $\Sigma^{orb}(X)$ is empty, then the definition of an orbifold reduces to the usual definition of smooth (or complex) manifold.

Definition 1.5. The order $ord(X)$ of an orbifold $\mathcal{X} = (X, U)$ is the least common multiple of orders $ord(x)$, when it exists (when, for example, $X$ is compact). Otherwise the orbifold $\mathcal{X}$ is said to have infinite order.

Definition 1.6. An orbifold $\mathcal{X} = (X, U)$ is called cyclic if for every orbifold chart $(\tilde{U}, \Gamma, \phi)$ we have that $\Gamma = \mathbb{Z}/m\mathbb{Z}$, for some positive integer $m$.

After having defined the objects of the orbifold category, we have to define the morphisms between such objects.

Definition 1.7. Let $\mathcal{X} = (X, U)$ and $\mathcal{Y} = (Y, V)$ be two orbifolds. Then, a map $f : X \to Y$ is said to be smooth (or holomorphic) if for every $x \in X$ there exist orbifold charts $(\tilde{U}, \Gamma, \phi)$ about $x$ and $(\tilde{V}, \Delta, \psi)$ about $f(x)$ such that $f$ maps from $\tilde{U} = \phi(\tilde{U})$ to $V = \psi(\tilde{V})$ and lifts to a smooth (or holomorphic) map $\tilde{f} : \tilde{U} \to \tilde{V}$ satisfying $\psi \circ \tilde{f} = f \circ \phi$. Moreover, $\mathcal{X}$ and $\mathcal{Y}$ are said to be diffeomorphic (or biholomorphic) if there exist two smooth (or holomorphic) maps $\alpha : X \to Y$ and $\beta : Y \to X$ such that $\alpha \circ \beta = \mathbb{1}_Y$ and $\beta \circ \alpha = \mathbb{1}_X$.

We conclude this section with the following important remark, which will be used later in the chapter.

Remark 1.4. Given an orbifold $\mathcal{X} = (X, U)$, let us consider how the charts are glued together to yield the space $X$. Given $(\tilde{U}, \Gamma, \phi)$ and $(\tilde{V}, \Pi, \psi)$ with $x \in U \cap V$, there is by definition a third chart $(\tilde{W}, \Lambda, \mu)$ and embeddings $\lambda_1, \lambda_2$ from this chart into the other two. Here $W$ is an open set $x \in W \subset U \cap V$. These embeddings give rise to diffeomorphisms $\lambda_1^{-1} : \lambda_1(\tilde{W}) \to \tilde{W}$ and $\lambda_2 : \tilde{W} \to \lambda_2(\tilde{W})$, which can be composed to provide an equivariant diffeomorphism $\lambda_2 \circ \lambda_1^{-1}$ between an open set in $\tilde{U}$ and an open set in $\tilde{V}$, regarded as $\Lambda$-spaces. We can then proceed to glue $U/\Gamma$ with $V/\Pi$ according to the induced homeomorphism of subsets, i.e., identify $\phi(\tilde{u}) \sim \psi(\tilde{v})$ if $\lambda_2 \lambda_1^{-1}(\tilde{u}) = \tilde{v}$. Now, define

$$Y = \bigsqcup_{U \in U} (U/\Gamma)/\sim$$

which is the space obtained by performing these identifications on the orbifold atlas. Moreover, the maps $\phi : \tilde{U} \to X$ glue together to give a homeomorphism $\Phi : Y \to X$. Thus, we have recovered the underlying space $X$ of the orbifold $\mathcal{X}$ from the orbifold charts. Sometimes, we will call the maps $\lambda_{12} = \lambda_2 \lambda_1^{-1}$ transition functions.

### 1.2 Examples

Now that we have given the basic definitions, it is worth showing some examples of construction of orbifolds. A standard and, perhaps, the most natural way to construct an orbifold is given by the following
Proposition 1.1. Let $M$ be a manifold acted smoothly and properly discontinuously by a group $\Gamma$. Then the quotient space $M/\Gamma$ is naturally endowed with an orbifold structure.

Proof. Consider a point $p \in M$. Then the isotropy subgroup $\Gamma_p \subset \Gamma$ is finite since the action is properly discontinuous. Moreover, there is an open neighbourhood $\tilde{U}_p$ such that $\gamma \tilde{U}_p \cap \tilde{U}_p = \emptyset$ if $\gamma \notin \Gamma_p$ and $\gamma \tilde{U}_p \cap \tilde{U}_p = \tilde{U}_p$ if $\gamma \in \Gamma_p$. Then the natural projection to the quotient $\phi : \tilde{U}_p \to \tilde{U}_p/\Gamma_p = U_p$ is a local homeomorphism, invariant under the action of $\Gamma_p$. We construct the cover $\{U_p\}$ for $M/\Gamma$ and local uniformizing systems of the form $\{\tilde{U}, \Gamma, \phi\}$ by adding when necessary finite intersections of the form $U_{p_1} \cap \cdots \cap U_{p_n}$ to guarantee that condition 2. in Definition 1.2 holds. The quotient $M/\Gamma$, which is always Hausdorff, is then an orbifold. \qed

The previous proposition gives us a way to construct a large quantity of examples of orbifolds.

Example 1.1. Let $X = \mathbb{T}^4/(\mathbb{Z}/2\mathbb{Z})$, where the action is generated by the involution $\tau$ defined by

$$\tau(e^{ilt_1}, e^{ilt_2}, e^{ilt_3}, e^{ilt_4}) = (e^{-ilt_1}, e^{-ilt_2}, e^{-ilt_3}, e^{-ilt_4}).$$

This orbifold is called the Kummer surface and it can be shown that it has sixteen singular points.

Example 1.2. Let us consider the six dimensional torus $\mathbb{T}^6$ seen as the quotient of $\mathbb{C}^3$ by the lattice $\Gamma$ of integral points. Consider $(\mathbb{Z}/2\mathbb{Z})^2$ acting on $\mathbb{T}^6$ via a lifted action on $\mathbb{C}^3$, where the generators $\sigma_1$ and $\sigma_2$ act as follows:

$$\sigma_1(z_1, z_2, z_3) = (-z_1, -z_2, z_3),$$
$$\sigma_2(z_1, z_2, z_3) = (-z_1, z_2, -z_3),$$
$$\sigma_1 \sigma_2(z_1, z_2, z_3) = (z_1, -z_2, -z_3).$$

Our example is $X = \mathbb{T}^6/(\mathbb{Z}/2\mathbb{Z})^2$. This example was considered by Vafa and Witten.

Another important example we are going to mention comes from the interplay between Algebraic Geometry and Theoretical Physics, which turns out to be an incredible source of examples of orbifold.

Example 1.3. Firstly consider the suset $Y \subset \mathbb{CP}^4$ given by the homogeneous equation

$$z_0^5 + z_1^5 + z_2^5 + z_3^5 + tz_0z_1z_2z_3z_4 = 0,$$

where $t \in \mathbb{C}$ is a constant parameter. So, $Y$ is a degree five hypersurface of $\mathbb{CP}^4$ and it admits an action of $(\mathbb{Z}/5\mathbb{Z})^3$. Indeed, let $\lambda$ be a fifth primitive root of unity and let the generator of the group $e_1, e_2$ and $e_3$ act as follows:

$$e_1(z_0, z_1, z_2, z_3, z_4) = (\lambda z_0, z_1, z_2, z_3, \lambda^{-1}z_4),$$
$$e_2(z_0, z_1, z_2, z_3, z_4) = (z_0, \lambda z_1, z_2, z_3, \lambda^{-1}z_4),$$
$$e_3(z_0, z_1, z_2, z_3, z_4) = (z_0, z_1, \lambda z_2, z_3, \lambda^{-1}z_4).$$

The quotient $X = Y/(\mathbb{Z}/5\mathbb{Z})^3$ is the celebrated mirror quintic.
1.2 Examples

Example 1.4. Suppose that $M$ is a complex manifold. One can consider the natural action of the symmetric group $S_n$ on the product $M^n$ and define the quotient space $X_n = M^n / S_n$. $S_n$ acts on $M^n$ by permuting coordinates. Tuples of points have isotropy according to how many repetitions they contain, with the diagonal being the fixed point set. $X_n$ has the natural structure of orbifold, given by Proposition 1.1. Moreover, if $\dim M = 1$, i.e. if $M$ is a Riemann Surface, then $X_n$ is a smooth manifold.

Example 1.5. Let $G \subset GL(n, \mathbb{C})$ be a finite subgroup and let $X = \mathbb{C}^n / G$. This is a major example of complex orbifold, which has the additional structure of an algebraic variety arising from the algebra of $G$-invariant polynomials on $\mathbb{C}^n$, by which we mean that one can view $X$ as $\text{Spec}(\mathbb{C}[x_1, \ldots, x_n]^G)$. This is a singular complex manifold with quotient singularity. Furthermore, if $G \subset SL(n, \mathbb{C})$, $X$ is said to be Gorenstein.

Example 1.6. A fundamental class of examples that are not hard to describe is given by orbifold Riemann Surfaces. Indeed, let $C$ be a Riemann Surface with $n$ marked points $p_1, \ldots, p_n$. Given such data, we can endow $C$ with an orbifold structure by specifying the singularity at the points $p_1, \ldots, p_n$: around $p_i$, an orbifold chart is given by $D \to D/\mathbb{Z}_{m_i}$, where $D$ is a small disk around zero in the complex plane $C$ and the action of the cyclic group of order $m_i$ is generated by multiplication for a primitive $m_i$-th root of unity, i.e. $q.z = \lambda z$, where $\mathbb{Z}_{m_i} = \langle \varphi \rangle$ and $\lambda = 1$.

Suppose that an orbifold Riemann Surface $\Sigma$ has genus $g$ and $k$ marked points. Thurston [34] has shown that it is a global quotient if and only if $g + 2k \geq 3$ or $g = 0$ with $m_1 = m_2$. In any case an orbifold Riemann Surface is always a quotient orbifold, as it can be expressed as $X^3 / S^1$, where $X^3$ is a 3-manifold called a Seifert fiber manifold (see [23]).

Finally, we describe weighted projective spaces. These are in fact the examples we are most interested in and are our first example of orbifolds which is not a global quotient, i.e. their orbifold structure is not the one described in Proposition 1.1. Indeed, in a later section, we will prove that any polarised cyclic orbifold can be embedded into a weighted projective space, a property that will turn to be fundamental in the definition of orbifold $K$-(semi)stability.

Example 1.7. Fix a weight vector $w = (w_0, \ldots, w_n)$ of positive integers and consider the group $\mathbb{C}^*$ of non-zero complex numbers acting on $\mathbb{C}^{n+1}$ as follows:

$$\lambda.(z_0, \ldots, z_n) = (\lambda^{w_0}z_0, \ldots, \lambda^{w_n}z_n), \quad \lambda \in \mathbb{C}^*, z_i \in \mathbb{C}.$$

Thus, we define the weighted projective space with weight $w$ as the quotient space $(\mathbb{C}^{n+1} \setminus \{(0, \ldots, 0)\}) / \mathbb{C}^*$ and we denote it as $\mathbb{C}P(w)$. Note that, when $w = (1, \ldots, 1)$, this is the usual $n$-dimensional projective space. Keeping this in mind, it is easy to give an orbifold atlas for $\mathbb{C}P(w)$. Firstly, consider the open sets given by

$$U_i = \{ [z_0, \ldots, z_n] \in \mathbb{C}P(w), z_i \neq 0 \}, i = 0, \ldots, n.$$
Clearly their union is \( \mathbb{CP}(w) \). Secondly, we take \( \tilde{U}_i = \mathbb{C}^n, i = 0, \ldots, n, \Gamma_i = \mathbb{Z}_{w_i} \) and define \( \phi_i : \tilde{U}_i \to U_i \) by \( \phi_i(z_1, \ldots, z_n) = [z_1, \ldots, 1, \ldots, z_n] \). Moreover, we describe the action of \( \Gamma_i = \langle \lambda_i \rangle \) on \( \tilde{U}_i \) as follows:

\[
\lambda_i(z_0, \ldots, z_{n-1}) = (2\pi \frac{w_i}{w_0} \sqrt{-1} z_0, \ldots, e^{2\pi \frac{w_i}{w_0} \sqrt{-1}} z_{i-1}, e^{2\pi \frac{w_i}{w_0} \sqrt{-1}} z_i, \ldots, e^{2\pi \frac{w_i}{w_0} \sqrt{-1}} z_{n-1}).
\]

It is then a straightforward computation to see that the induced map \( \tilde{U}_i/\Gamma_i \to U_i \) is a homeomorphism, when both spaces are endowed with the quotient topology.

Furthermore, we give a coordinate free approach to the construction of \( \mathbb{CP}(w) \), which will be very useful when dealing with the orbifold Kodaira embedding theorem. To this purpose, consider a graded complex vector space, by which we mean a vector space \( V \), endowed with a weight \( w \in \mathbb{Z} \). The \( w \)-th graded component is \( V^w = \bigoplus_i V_i \). Define \( \mathbb{WP}(V) = (V \setminus \{0\})/\mathbb{C}^* \). The weights, so the number of \( a_j \) that equal \( i \) is \( \dim V^i \).

### 1.3 Orbisheaves and Orbibundles

Our next step is to define and describe sheaves and bundles on orbifolds. As we will immediately see, these concepts are almost straightforward generalisations of the ones we are used to consider in the context of smooth (or complex) manifolds. Indeed, we begin with the following

**Definition 1.8.** Let \( X = (X, \mathcal{U}) \) be an orbifold. A sheaf \( F \) on \( X \) or orbisheaf on \( X \) consists of a sheaf \( F_{\tilde{U}} \) over \( \tilde{U} \) for each orbifold chart \( (\tilde{U}, \Gamma, \phi) \) such that for each injection \( \lambda : (\tilde{U}, \Gamma, \phi) \to (\tilde{V}, \Pi, \psi) \) there is an isomorphism of sheaves \( F(\lambda) : F_{\tilde{U}} \to \lambda^* F_{\tilde{V}} \), which is functorial in \( \lambda \).

**Remark 1.5.** Let \( F \) be an orbisheaf on \( X \) and \( (\tilde{U}, \Gamma, \phi) \) an orbifold chart. By Remark 1.2 we know that each \( \gamma \in \Gamma \) defines an injection \( \lambda_{\gamma} \), so there is a sheaf map \( F(\gamma) : F_{\tilde{U}} \to \gamma^* F_{\tilde{U}} \). Let \( s \in F_{\tilde{U}, x} \) be a point in the stalk over \( x \in \tilde{U} \). Then \( F(\gamma)(s) \) lies in the stalk over \( \gamma^{-1} x \). This defines an action of \( \Gamma \) on the sheaf \( F_{\tilde{U}} \), which says that \( F_{\tilde{U}} \) is a \( \Gamma \)-equivariant sheaf over \( \tilde{U} \). So every orbisheaf \( F \) is equivariant under the local uniformizing groups \( \Gamma_i \).

**Example 1.8.** Let \( X \) be an orbifold. We define the **structure sheaf** \( O_X \) to be the orbisheaf defined by the structure sheaf \( O_{\tilde{U}} \) on each orbifold chart \( (\tilde{U}, \Gamma, \phi) \), by which we mean that \( O_{\tilde{U}} \) is the sheaf of \( \mathcal{C}^\infty \) functions on \( \tilde{U} \) for real orbifolds and the sheaf of holomorphic functions on \( \tilde{U} \) for complex orbifolds. Note that \( O_X \) is well-defined since each injection \( \lambda : (\tilde{U}, \Gamma, \phi) \to (\tilde{U}', \Gamma', \phi') \) induces an isomorphism \( O_{\tilde{U}} \cong \lambda^* O_{\tilde{U}'} \) by sending \( f \in O_{\tilde{U}, x} \) to \( f \circ \lambda^{-1} \in (\lambda^* O_{\tilde{U}'})_x \). Note that \( O_X \) is a sheaf of commutative rings on each local uniformizing neighbourhood \( \tilde{U}_i \).

We define now morphisms of orbisheaves in such a way that they generalize morphisms of sheaves just as orbisheaves generalize sheaves. Indeed, we have the following
Definition 1.9. A morphism of orbisheaves $\alpha : \mathcal{F} \to \mathcal{F}'$ is a family of sheaf morphisms $\alpha_{\tilde{U}} : \mathcal{F}_{\tilde{U}} \to \mathcal{F}'_{\tilde{U}}$, one for each orbifold chart, that is compatible with injections $\lambda$, in the sense that the following diagram

$$
\begin{array}{ccc}
\mathcal{F}_{\tilde{U}} & \xrightarrow{\alpha_{\tilde{U}}} & \mathcal{F}'_{\tilde{U}} \\
\mathcal{F}(\lambda) \downarrow & & \downarrow \mathcal{F}'(\lambda) \\
\lambda^*\mathcal{F}_{\tilde{V}} & \xrightarrow{\lambda^*\alpha_{\tilde{V}}} & \lambda^*\mathcal{F}'_{\tilde{V}}
\end{array}
$$

commutes.

Remark 1.6. By the above definitions, it can be easily seen that the collection of all orbisheaves of sets, rings, abelian groups, etc. forms a category, called the category of orbisheaves on $X$.

We now introduce two important notions, which are the orbifold counterpart of locally free sheaves and vector bundles on smooth manifolds.

Definition 1.10. An orbisheaf $\mathcal{M}$ of $O_X$-modules on $X$ is said to be locally $V$-free or simply locally free if each point $x \in X$ has an orbifold local chart such that $\mathcal{M}_{\tilde{U}} \cong (O_{\tilde{U}})^r$ for some positive integer $r$, called the rank of $\mathcal{M}$. An orbisheaf of rank 1 is called invertible.

Definition 1.11. An orbibundle, or $V$-bundle, over an orbifold $X = (X, \mathcal{U})$ consists of a fibre bundle $B_{\tilde{U}}$ over $\tilde{U}$ for each orbifold chart $(\tilde{U}, \Gamma, \phi_i) \in \mathcal{U}$ with Lie group $G$ and fibre $F$ a smooth $G$-manifold which is independent of $\tilde{U}$, together with homomorphisms $h_{\tilde{U}} : \Gamma_i \to G$ satisfying:

1. if $b$ lies in the fibre over $\tilde{x}_i \in \tilde{U}_i$ then, for each $\gamma \in \Gamma_i$, $h_{\tilde{U}}(\gamma).b$ lies in the fibre over $\gamma^{-1}\tilde{x}_i$.

2. if the map $\lambda_{ji} : \tilde{U}_i \to \tilde{U}_j$ is an injection, then there is a bundle map $\lambda_{ji}^* : B_{\tilde{U}_j(\lambda_{ji}(\tilde{U}_i))} \to B_{\tilde{U}_i}$, satisfying the condition that if $\gamma \in \Gamma_i$ and $\gamma' \in \Gamma_j$ is the unique element such that $\lambda_{ji} \circ \gamma = \gamma' \circ \lambda_{ji}$, then $h_{\tilde{U}_j}(\gamma) \circ \lambda_{ji}^* = h_{\tilde{U}_i}(\gamma')$, and if $\lambda_{kj} : \tilde{U}_j \to \tilde{U}_k$ is another such injection then $(\lambda_{kj} \circ \lambda_{ji})^* = \lambda_{ji}^* \circ \lambda_{kj}^*$.

If the fibre $F$ is a vector space of dimension $r$ and $G$ acts on $F$ as linear transformation of $F$, then the $V$-bundle is called a vector $V$-bundle of rank $r$.

Remark 1.7. Both the definitions of orbisheaves and orbibundles consist of a sequence of sheaves or bundles defined on the disjoint union $\tilde{U} = \bigsqcup \tilde{U}_i$ of the local uniformizing neighbourhoods that satisfy certain compatibility conditions with respect to the action of the local uniformizing groups and the injections.

Having defined the notion of orbibundle, what we next have to describe is obviously the concept of section of such an orbibundle. To this purpose, we have the following...
Remark 1.8. Given local sections \( s_\gamma \) of the bundle \( B_\Gamma \) for each orbifold chart \( (\tilde{U}, \Gamma, \phi) \) such that \( \tilde{U} \subseteq W \) and for any \( \tilde{x} \in \tilde{U} \) we have

1. for each \( \gamma \in \Gamma \), \( s_\gamma(\gamma^{-1}\tilde{x}) = h_\Gamma(\gamma)s_\Gamma(\tilde{x}) \),

2. if \( \lambda : (\tilde{U}, \Gamma, \phi) \to (\tilde{U}', \Gamma', \phi') \) is an injection, then \( \lambda^*s_\Gamma(\lambda(x)) = s_\Gamma(\tilde{x}) \).

If each of the local sections \( s_\gamma \) is continuous, smooth, holomorphic, etc., then we say that \( s \) is continuous, smooth, holomorphic, etc., respectively.

Example 1.9. Let \( \mathcal{E} \) be an orbibundle over an orbifold \( \mathcal{X} \). Then a section \( s \) of \( \mathcal{E} \) over an open set \( W \subseteq X \) is given by sections \( s_\gamma \) of the bundle \( B_\gamma \) for each orbifold chart \( (\tilde{U}, \Gamma, \phi) \) such that \( \tilde{U} \subseteq W \) and for any \( \tilde{x} \in \tilde{U} \) we have

\[ s^I = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} s_\gamma \circ \gamma. \]

Note that this determines a well-defined map from the underlying space, namely: \( s_U : U = \phi(\tilde{U}) \to B_\Gamma \). Using invariant local sections over each orbifold chart we can obtain global invariant sections. Then, we view objects on an orbifod interchangeably as invariant objects on \( \tilde{U} \) or objects on \( U \).

Now, it is worth noting that any orbibundle \( \mathcal{E} \) over an orbifold \( \mathcal{X} \) has a total space which inherits an orbifold structure. For sake of simplicity, we will give an explicit construction only in the case of the orbifold tangent bundle \( T\mathcal{X} \) over a real orbifold \( \mathcal{X} \).

Definition 1.12. Let \( \mathcal{E} \) be an orbibundle over an orbifold \( \mathcal{X} \). Then a section \( s \) of \( \mathcal{E} \) over an open set \( W \subseteq X \) is given by sections \( s_\gamma \) of the bundle \( B_\gamma \) for each orbifold chart \( (\tilde{U}, \Gamma, \phi) \) such that \( \tilde{U} \subseteq W \) and for any \( \tilde{x} \in \tilde{U} \) we have

\[ \text{for each } \gamma \in \Gamma, s_\gamma(\gamma^{-1}\tilde{x}) = h_\Gamma(\gamma)s_\Gamma(\tilde{x}), \]

\[ \text{if each } \lambda : (\tilde{U}, \Gamma, \phi) \to (\tilde{U}', \Gamma', \phi') \text{ is an injection, then } \lambda^*s_\Gamma(\lambda(x)) = s_\Gamma(\tilde{x}). \]

If each of the local sections \( s_\gamma \) is continuous, smooth, holomorphic, etc., then we say that \( s \) is continuous, smooth, holomorphic, etc., respectively.

Examples 1.9. Let \( \mathcal{X} = (X, \mathcal{U}) \) be a real orbifold. Given a chart \( (\tilde{U}, \Gamma, \phi) \), we consider the tangent bundle \( T\tilde{U} \); by assumption, \( \Gamma \) acts smoothly on \( \tilde{U} \) and hence it will also act smoothly on \( T\tilde{U} \). Indeed, if \( (\tilde{a}, \tilde{v}) \), is a typical element there, then \( \gamma.(\tilde{a}, \tilde{v}) = (\gamma \cdot \tilde{a}, D\gamma \tilde{a}(\tilde{v}) \). Furthermore, the projection \( T\tilde{U} \to \tilde{U} \) is equivariant with respect to this action, thus we obtain a natural projection \( p : T\tilde{U}/\Gamma \to U \) by composing with the map \( \phi \). Then, if \( x = \phi(\tilde{x}) \) we have

\[ p^{-1}(x) = \{ \Gamma(z, v) | z = \tilde{x} \} \subseteq T\tilde{U}/\Gamma. \]

We claim that this fibre is homeomorphic to \( T\Gamma_x \). Indeed, define \( f : p^{-1}(x) \to T\Gamma_x \) by \( f(\gamma(\tilde{x}, v)) = \Gamma_x v \). Then, it is easy to see that \( f \) is both well defined and injective. Since continuity and surjectivity are clear, we get our claim. So, we have constructed a bundle-like object where the fibre is no longer a vector space, but rather a quotient of the form \( \mathbb{R}^n / \Gamma_0 \), where \( \Gamma_0 \subseteq GL(n, \mathbb{R}) \) is a finite group.

Now, to construct the orbifold tangent bundle \( T\mathcal{X} \) we simply need to glue together the bundles defined over the charts. The resulting space is an orbifold, with an atlas \( \mathcal{T} \mathcal{U} \) compromising local charts \( (T\tilde{U}, \Gamma, \pi) \) over \( T\mathcal{U} \). Indeed, define \( f : p^{-1}(x) \to T\Gamma_x \) by \( f(\gamma(\tilde{x}, v)) = \Gamma_x v \). Then, it is easy to see that \( f \) is both well defined and injective. Since continuity and surjectivity are clear, we get our claim. So, we have constructed a bundle-like object where the fibre is no longer a vector space, but rather a quotient of the form \( \mathbb{R}^n / \Gamma_0 \), where \( \Gamma_0 \subseteq GL(n, \mathbb{R}) \) is a finite group.

Keeping in mind Remark 1.4, we note that the transition functions \( \lambda_{12} = \lambda_2 \lambda_1^{-1} \) are smooth, so we can use the transition functions \( D\lambda_{12} : T\lambda_1(\tilde{W}) \to T\lambda_2(\tilde{W}) \) in order to glue \( T\Gamma_x \) to \( U \) with \( T\Gamma_v \to V \). Thus, we define the space \( TX \) as the quotient space \( \bigsqcup_{U \in \mathcal{U}} (T\Gamma_x) \). The previous arguments then prove that the orbifold tangent bundle
\( T\mathcal{X} = (TX, T\mathcal{U}) \) of an \( n \)-dimensional orbifold \( \mathcal{X} \) has the structure of a \( 2n \)-dimensional orbifold.

The cotangent orbibundle and the tensor bundles are constructed similarly. Thus, one can easily construct Riemannian metrics, symplectic 2-forms, connections, etc., such that the injections are maps of the corresponding \( \Gamma \)-structure. We have then, for example,

**Definition 1.13.** Let \( \mathcal{X} = (\mathcal{X}, \mathcal{U}) \) denote an orbifold with tangent bundle \( T\mathcal{X} \).

- We call a non-degenerate symmetric section \( g \) of \( T^*\mathcal{X} \otimes T^*\mathcal{X} \) a Riemannian metric on \( \mathcal{X} \), i.e. a Riemannian metric \( g_i \) on each local uniformizing neighbourhood \( \tilde{U}_i \), that is invariant under the local uniformizing groups \( \Gamma_i \) and the injections \( \lambda_{ji} : (\tilde{U}_i, \Gamma_i, \phi_i) \to (\tilde{U}_j, \Gamma_j, \phi_j) \) are isometries, i.e. \( \lambda^*_{ji}(g_j|_{\lambda_{ji}(\tilde{U}_i)}) = g_i \).

- we define a differential \( k \)-form as a section of \( \wedge^k T\mathcal{X} \); the exterior derivative is defined locally as for manifolds in the usual way. Hence we can define the de Rham cohomology \( H^*(\mathcal{X}) \);

- an almost complex structure on \( \mathcal{X} \) is an endomorphism \( J : T\mathcal{X} \to T\mathcal{X} \) such that \( J^2 = -1 \);

- a symplectic structure on \( \mathcal{X} \) is a non-degenerate closed 2-form.

- if \( \mathcal{X} \) is a complex orbifold, then the Dolbeaut cohomology is defined in the usual way.

**Remark 1.9.** Similarly, we can define hermitian metrics on complex orbifolds and, with a slight modification of usual partition of unity argument, one can prove that any (complex) orbifold admits a (Hermitian) Riemannian metric.

**Remark 1.10.** Integration theory also goes through. In particular, if \( V \) is an open subset of \( \phi(U) \), where \( (U, \Gamma, \phi) \) is a fixed orbifold chart, then, given a (measurable) \( n \)-form \( \omega \), we define

\[
\int_V \omega = \frac{1}{|\Gamma|} \int_{\phi^{-1}(V)} \omega_{\tilde{U}}.
\]

Hence, integration on an orbifold is defined exactly as for smooth manifolds, with the only exception that we have to divide for the order of the local uniformizing \( \Gamma \). So all the standard integration techniques, such as Stoke’s Theorem, hold on orbifolds.

Following the aforementioned general principle for defining tensors and, more in general, sections of orbi-bundles, we say that a Kähler metric is a Riemannian metric which satisfies the Kähler condition on each orbifold chart.

**Example 1.10.** Consider \( \mathbb{C} \), on which \( \Gamma_m = \mathbb{Z} / m\mathbb{Z} \) acts by multiplication for a primitive \( m \)-th root of unity. Then, following Example 1.5, we get the quotient space \( \mathcal{X} = \mathbb{C} / \Gamma_m = \text{Spec}(\mathbb{C}[z]^{\Gamma_m}) = \text{Spec} \mathbb{C}[z^m] = \text{Spec} \mathbb{C}[w] = \mathbb{C} \). Clearly, the orbifold chart that we need
to describe the orbifold structure of $X$ is given by $C \ni z \mapsto z^m = w \in C$. The standard orbifold Kähler metric on $X$ is then given by $\frac{\sqrt{-1}}{2} dz \wedge d\bar{z}$, where $z$ is the coordinate on $C$ upstairs and $w$ the coordinate on $X$. Then, we see that, on the quotient $X$, the metric takes the form $\frac{\sqrt{-1}}{2} m^{-2} |w| \hat{z}^{-2} dw \wedge d\bar{w}$. Hence, this example shows how Kähler metrics can descend to the underlying space $X$, to define Kähler metrics there, but with possible singularities on the orbifold locus, which, in our case is just $\{0\}$.

**Example 1.11** (Local model of conical Kähler metrics). The previous example has a straightforward generalisation in higher dimensions: let us consider $C^n$ on which $\Gamma_m = \mathbb{Z} / m\mathbb{Z}$ acts by multiplication for a primitive $m$-th root of unity on the first coordinate $z_1$ and trivially on $z_2, \ldots, z_n$. Then the quotient space $X = C^n / \Gamma_m = \text{Spec}(C[z_1, \ldots, z_n]^\Gamma_m) = \text{Spec} C[z_1^n, z_2, \ldots, z_n] = \text{Spec} C[w, z_2, \ldots, z_n] = C^n$ inherits the orbifold structure, given by the orbichart $C^n \ni (z_1, z_2, \ldots, z_n) \mapsto (z_1^m = w, z_2, \ldots, z_n) \in C^n$. In particular, $X$ is an orbifold with cyclic quotient singularities in codimension one: indeed the singular locus of $X$ is $\Sigma(X) = \{ (0, z_2, \ldots, z_n) \mid z_1 \in C, i = 2, \ldots, n \} \cong C^{n-1}$. Thus, this is the local model of the orbifolds we are mostly interested in. Just as in Example 1.10, the standard Kähler metric on $X$ is given by $(\frac{\sqrt{-1}}{2})^n (dz_1 \wedge d\bar{z}_1 + \cdots + dz_n \wedge d\bar{z}_n)$. Then, on the quotient $X$, the metric takes the form $(\frac{\sqrt{-1}}{2})^n (m^{-2} |w| \hat{z}^{-2} dw \wedge d\bar{w} + dz_2 \wedge d\bar{z}_2 + \cdots + dz_n \wedge d\bar{z}_n)$. This is a conical metric with cone angle $\frac{2\pi m}{m}$, singular along $\Sigma(X)$.

**Remark 1.11.** The previous Example motivates our interest in looking for solutions to the cscK equation which are conical metrics singular along a divisor $D$ of a compact manifold $X$.

We conclude this section with a fundamental example of orbifold line bundle, whose importance will be clear later in the chapter when we will embed polarised cyclic orbifolds into weighted projective spaces.

**Example 1.12** (*Tautological and hyperplane line bundles on WP(V)*). This is the orbi-line bundle over the orbifold $WP(V)$ with fibre over $[v]$ the union of the orbit $C^* v$ and $0 \in V$. Note that this fibre can be given the vector space structure as follows: any two elements in a fibre can be written as $w_i = t_i v$ for $t_i \in C$, $i = 1, 2$ so the linear structure is defined as $aw_1 + bw_2 = (at_1 + bt_2) v$. The orbifold chart for $WP(V)$ described in Example 1.7 is also a trivializing chart for this orbifold line bundle. Indeed, over this chart, $O_{WP(V)}(-1)$ reduces to $C^n \times C$ with the weight one $Z_{\lambda_i}$ action on the line $C$ times by its action on $C^n$, also described in Example 1.7. Indeed, we have the map 
\[ C^n \times C \to C^{n+1}, \]
\[ (z_0, \ldots, z_n) \mapsto (t^{\lambda_1} z_0, \ldots, t^{\lambda_n} z_n), \]
which becomes $Z_{\lambda_i}$-equivariant when we use the action of Example 1.7 on $C^n$, the standard weight-one action on $C$ and the original weighted $C^*$-action on $C^{n+1}$. As in the case of smooth manifolds, the **hyperplane line bundle** $O_{WP(V)}(1)$ is defined as the dual line bundle of $O_{WP(V)}(-1)$, the fibres of which are endowed with the linear structure described above.
1.4 Complex Orbifolds and divisors

We will now focus our attention on complex orbifolds. In particular, we will consider cyclic complex orbifolds (see Definition 1.6), as it is for them that Ross and Thomas, in [20], could extend the notion of K-(semi)stability and prove the orbifold version of Donaldson’s theorem - in the next chapter we will give detailed definitions and statements.

Firstly, we begin with characterizing the structure of the orbifold charts in the case of cyclic stabiliser. Consider then a point in the orbifold locus with stabiliser group $\mathbb{Z}_m$ and let $p$ be its preimage in an orbifold chart and $m_p$ be its maximal ideal. We can split the cotangent space $m_p/m_p^2$ into weight spaces under the group action and, by using the fact that the ring of formal power series about $p$ is the space $\bigoplus_i S_i (m_p/m_p^2)$, we see that, locally in the analytic topology, there is a chart $\tilde{U} \to \tilde{U}/\mathbb{Z}_m$ of the form $$(z_1, \ldots, z_n) \mapsto (z_1^{a_1}, z_2^{a_2}, \ldots, z_k^{a_k}, z_{k+1}, \ldots, z_n),$$ for some integers $a_i$ which divide $m$. This means that, locally, orbifold chart in the cyclic case are generalizations of those in Example 1.10.

Next, we want to analyse more in depth the structure of the underlying space of a complex orbifold. To this purpose, fix a complex orbifold $X = (X, \mathcal{U})$. We can endow $X$ with the structure of locally $\mathbb{C}$-ringed space $(X, O_X)$ as follows: the structure sheaf $O_X$ has stalks $O_x$ for $x \in U = \phi(\tilde{U}) \subset X$ that are isomorphic to the local ring $O_{\Gamma C}^n$ of germs of $\Gamma$-invariant holomorphic functions, where $\Gamma$ is a finite group. In particular, if $x \in X$ is a regular point, i.e. $x \in X \setminus \Sigma(X)$, since the local uniformizing groups at such a point are trivial, then $O_x$ is isomorphic to the ring $O_{\mathbb{C}^n}$ of convergent power series in $\mathbb{C}^n$; if, instead, $x \in \Sigma(X)$ is a singular point, the ring $O_x$ is isomorphic to the ring $O_{\Gamma C}^n$. Actually, one can prove a much stronger result concerning the structure of the underlying space of a complex orbifold, for whose proof we refer to [4] and [12].

**Proposition 1.2.** The locally ringed space $(X, O_X)$ associated to a complex orbifold has the following properties:

1. $(X, O_X)$ is a reduced normal complex space,
2. the singular locus $\Sigma(X)$ is a closed reduced complex subspace of $X$,
3. the smooth locus $X_{\text{reg}}$ is a complex manifold and a dense open subset of $X$.

Our interest in this proposition relies on the fact that the result can be used as the starting point for an alternative definition of the concept of cyclic complex orbifold, seen as a log pair $(X, \Delta)$, where $X$ is a variety and $\Delta$ a $\mathbb{Q}$-divisor of $X$. Notice that this is the point of view adopted in [5] and [12]. Indeed, we start with a normal, compact, complex space $X$. And, as before, we say that $X$ is a complex orbifold if it is locally given by charts written as quotients of smooth coordinate charts, i.e. $X$ can be covered by
open charts $X = \bigcup U_i$ and for each $U_i$ there is a smooth complex space $\tilde{U}_i$ and a finite group $\Gamma_i$ acting on $\tilde{U}_i$ such that $U_i$ is biholomorphic to the quotient space $\tilde{U}_i/\Gamma_i$. The classical, or well formed, case is when the fixed point set of every non-identity element of every group $\Gamma_i$ has codimension at least two. In this case $X$ alone determines the orbifold structure.

However, as mentioned before, the most interesting case for us is the one in which there are codimension one fixed point sets. Let us fix then a quotient map $\phi_i : \tilde{U}_i \to U_i$; it has branch divisors $D_{ij} \subset U_i$ and ramification divisors $R_{ij} \subset \tilde{U}_i$. Let $m_{ij}$ denote the ramification index over $D_{ij}$. Then, locally near a general point of $R_{ij}$ the map $\phi_i$ looks like

$$C^n \to C^n, \quad \phi_i(z_1, \ldots, z_n) = (x_1 = z_1^{m_{ij}}, x_2 = z_2, \ldots, x_n = z_n).$$

Consequently, we have the following pullback equality,

$$\phi_i^* (dx_1 \wedge \cdots \wedge dx_n) = m_{ij} z_1^{m_{ij}} dz_1 \wedge \cdots \wedge dz_n.$$

The compatibility condition between the charts that has to be assumed is that there are global divisors $D_j \subset X$ and ramification indexes such that $D_{ij} = U_i \cap D_j$ and, possibly after re-indexing, $m_{ij} = m_j$. All these data can be presented in a unified way with a single $\mathbb{Q}$-divisor, called the branch divisor of the orbifold,

$$\Delta = \sum (1 - \frac{1}{m_j}) D_j.$$

The main fact is that we may identify an orbifold with the pair $(X, \Delta)$, since it turns out that the orbifold structure is uniquely determined by this log pair. We will show this in the next example, when $X$ is a complex manifold.

**Example 1.13 (Orbifolds as log pairs).** Let $X$ be a complex manifold and $D = \sum_{i \in I} D_i$ a divisor with local normal crossing, by which we mean that for any point $x \in X$ there is a holomorphic coordinate system $(V, z_1, \ldots, z_n)$ such that $D \cap V = \{ z \in V \mid z_1 \cdots z_k = 0 \}$. If $D_i \cap V \neq \emptyset$ then $D_i \cap V$ is the union of some of the hypersurfaces $\{ z_j = 0 \}$. Note that $D$ is said to be a divisor with global normal crossing if, in addition, each $D_i$ is smooth. Moreover, for any $i \in I$ fix an integer $m_i > 1$ and define $\Delta = \sum (1 - \frac{1}{m_i}) D_i$. Then, $(X, \Delta)$ is an orbifold. Indeed, fix a coordinate system as above and put $m'_j = m_i$ if $\{ z_j = 0 \} \subset D_i \cap V$ and define

$$\phi : \tilde{U} \to V, \quad \phi(z_1, \ldots, z_n) = (z_1^{m'_i}, \ldots, z_k^{m'_k}, z_{k+1}, \ldots, z_n).$$

Also set $\Gamma = \mathbb{Z}_{\gcd(m'_1, \ldots, m'_k)}$ acting as usual on $\tilde{U}$. Then, the triple $(\tilde{U}, \Gamma, \phi)$ is an orbifold chart for $(X, \Delta)$ and the orbifold structure defined in this way is uniquely determined.

Summarizing the results described in this section, we have that a complex orbifold with cyclic quotient singularities in codimension one is described by the pair $(X, \Delta)$, where

- $X$ is a smooth variety,
1.5 Orbifold Kodaira Embedding Theorem

- \( \Delta \) is a \( Q \)-divisor of the form \( \Delta = \sum_i (1 - \frac{1}{m_i}) D_i \),
- the \( D_i \) are distinct smooth irreducible effective divisors,
- \( D = \sum_i D_i \) has normal crossing, and
- the \( m_i \) are positive integers such that \( m_i \) and \( m_j \) are coprime if \( D_i \) and \( D_j \) intersect.

Then the stabiliser group of points in the intersection of several components \( D_i \) will be the product of groups \( \mathbb{Z}_{m_i} \), and this is cyclic by the coprimality assumption.

This alternative point of view has obviously a certain number of advantages: firstly, to consider log pairs means to consider smooth objects with the extra datum given by the orbifold divisor \( \Delta \). Moreover, we can relate the usual divisor classes representing fundamental line bundles on the underlying variety with their orbifold counterparts in term of \( \Delta \). This is easily seen in the following example.

**Example 1.14.** Consider a smooth variety \( X \) and a divisor \( D \) on \( X \) on which we put the stabiliser group \( \mathbb{Z}/m\mathbb{Z} \). Locally, we can find a coordinate \( x \) such that \( D \) can be written as the zeros locus of \( x \), i.e., locally, \( D = \{x = 0\} \). Furthermore, we choose a coordinate chart \( z \) such that \( x = z^m \). In this setting, we have that \( dx = mz^{m-1}dz = mx^{1-\frac{1}{m}}dz \) and this shows that \( X \) has orbifold canonical bundle

\[
K_{orb} = K_X + \left(1 - \frac{1}{m}\right) D = K_X + \Delta,
\]

where \( K_X \) is the canonical divisor of the underlying variety \( X \). More generally, if the orbifold locus is a union of divisors \( D_i \) with stabiliser groups \( \mathbb{Z}_{m_i} \), then \( K_{orb} = K_X + \Delta \), where \( \Delta = \sum_i (1 - \frac{1}{m_i}) D_i \).

1.5 Orbifold Kodaira Embedding Theorem

In this section we will address the problem of extending the celebrated Kodaira Embedding Theorem to the context of orbifolds. To arrive to such extension we have to go through some steps. First of all, we need to precisely specify the setting which we want to work in and to this purpose it might be useful to remind the notion of ampleness of a line bundle on a complex manifold.

**Definition 1.14.** Let \( L \) be a line bundle on a complex manifold \( X \) of dimension \( n \). Then, we say that \( L \) is very ample if the map to the complete linear system \( \mathbb{P}(H^0(X, L)^*) \) associated to \( L \) is an embedding. We say that \( L \) is ample if \( L \otimes k \) is very ample for \( k \gg 0 \).

To extend the concept of ampleness of a line bundle to the context of orbifolds we have to keep in mind our main purpose, which is to embed such spaces into weighted projective space, taking into account that we want to "embed" the orbifold structure as well. Given this, we will see in the next example what kind of orbi-line bundles we are not considering as examples of ample line bundles.
Example 1.15. Consider $C$ acted by $\mathbb{Z}_2$ as usual: $z \mapsto -z$. Then, consider the orbifold $X = C / \mathbb{Z}_2$ given by taking the quotient space, with $z$ as coordinate upstairs and $x = z^2$ as coordinate downstairs on the quotient thought as a manifold. It follows that the space of invariant sections of any line bundle pulled back from $X$ is $C[z] = C[z^2]$. This means that we are missing the extra functions of $\sqrt{x} = z$, i.e. the quotient is considered only as a manifold, without the orbifold structure. We do not consider this as an example of ample orbi-line bundle, because if we tried to embed the quotient with its section we would contract the stabilisers, "losing" the orbifold structure.

We thus have the following

Definition 1.15. An orbifold line bundle $L$ on a cyclic orbifold $X$ is locally ample if in an orbifold chart around $x \in X$, the stabiliser group acts faithfully on the line $L_x$. We say that $L$ is orbi-ample if it is locally ample and globally positive, by which we mean that $L^{\otimes \text{ord}(X)}$ is ample as in Definition 1.14 when thought of as a line bundle on the underlying space $X$. By a polarised orbifold we mean a pair $(X, L)$ where $X$ is a cyclic orbifold and $L$ an orbi-ample line bundle on $X$.

Remark 1.12. Note that, from the Kodaira-Baily embedding Theorem [3], one can equivalently ask that $L$ admits a hermitian metric with positive curvature.

We are now able to clarify the set up in which we will prove the embedding theorem. To this purpose, fix a polarised orbifold $(X, L)$ and $k \gg 0$. Let $i$ run through a fixed indexing set $0, 1, \ldots, M$, where $M \geq \text{ord}(X)$, and let $V$ be the graded vector space $V = \bigoplus_i V^{i+k} := \bigoplus_i H^0(X, L^{k+i})^*$. The grading is given in such a way that the $i$-th summand has weight $k+i$. For a fixed $k$ we define a map $\phi_k : X \to \mathbb{P}(V)$, where, for brevity of notation, we denote $\mathbb{W}\mathbb{P}(V)$ with $\mathbb{P}(V)$. The map $\phi_k$ is defined as follows

$$\phi_k(x) = [\bigoplus_i \text{ev}_x^{k+i}].$$

Here we are fixing a trivialisation of $L_x$ on an orbifold chart, inducing trivialisations on all powers $L_x^{i+k}$ and then $\text{ev}_x^{k+i}$ is the element of the dual of $H^0(L^{k+i})$ which is just the evaluation map. The weights are chosen so that a change of trivialisation induces a change in $\text{ev}_x^{k+i}$ that differs only by the action of $\mathbb{C}^*$ on $V$ and this proves that the map is well defined at all points $x$ such that there exists a global section of some $L^{i+k}$ not vanishing at $x$.

Having clarified the working set-up, we are then able to prove the following

Theorem 1.1. If $(X, L)$ is a polarised orbifold, then for $k \gg 0$ the map (1.1) is an embedding of orbifolds, by which we mean that the orbifold structure on $X$ is pulled back from that on the weighted projective space $\mathbb{W}\mathbb{P}(V)$ and $\phi^* \mathcal{O}_{\mathbb{W}\mathbb{P}(V)}(1) \cong L$. 
Proof. Let us consider a point on the underlying space $x \in X$, which, by definition, has a stabiliser group of the form $\mathbb{Z}/m\mathbb{Z}$, $m \geq 1$, and local orbifold chart of the form $U \to U/(\mathbb{Z}/m\mathbb{Z})$. Let $y \in U$ (whose maximal ideal we call $m_y$) be a point which lies over $x$. We can decompose $m_y/m_y^2 = \bigoplus_l V_l$ into weight spaces. This means that, on $V_l$, $\mathbb{Z}/m\mathbb{Z}$ acts as $\lambda_l v = \lambda_l v$. Furthermore, since we have chosen the indexing set of $i$ to range over at least a full period of length $m$, at least one of the $L_i^k$ has weight 0: this is easily seen noting that if $\mathbb{Z}/m\mathbb{Z}$ acts on a vector space $V$ with weight $w$, then it acts on $V \otimes k$ with weight $wk$. Thus, for each $l$ there is at least one $i_l$ in the indexing set of $i$ such that $L_i^k \otimes V_i^l$ has weight 0.

Therefore each of these $\mathbb{Z}/m\mathbb{Z}$-modules has invariant local generators, defining local sections of the appropriate power of $L$ on $X$. For $k \gg 0$ these extend to global sections, by ampleness. Therefore, choosing a local trivialising chart about $y$, the sections of $L$ generate $O_y$ and $m_y/m_y^2$, so the pullback of the local functions on $WP(V)$ generate the local functions on $U$. It follows that the map is an embedding for large $k$.

Remark 1.13. Following the previous arguments, we can prove further results concerning the weighted Kodaira embedding introduced in this section. Indeed, let us begin with describing invariantly the map (1.1): any lift $\tilde{x} \in L_x^{-1}$ of $x$ is a linear functional on $L_x$. Similarly $\tilde{x} \otimes (k+i)^i$ is a linear functional on $L_x^{k+i}$. If we compose this linear functional with the evaluation map $ev^{k+i}_x: H^0(L_x^{k+i}) \to L_x^{k+i}$, we get

$$\tilde{x} \otimes (k+i) \circ ev^{k+i}_x: H^0(L_x^{k+i}) \to \mathbb{C}.$$ 

Therefore

$$\bigoplus_i (\tilde{x} \otimes (k+i) \circ ev^{k+i}_x) \in \bigoplus_i H^0(L_x^{k+i})^* = V$$

is a well defined point, i.e. it does not depend on the $\mathbb{C}^*$-action or the choices. This means that the map (1.1) lifts to a $\mathbb{C}^*$-equivariant embedding of the orbi-line

$$L_x^{-1} \hookrightarrow \bigoplus_i H^0(L_x^{k+i})^*.$$ 

This enables us to say that, under this weighted Kodaira embedding, we have that $\phi^*(O_{WP}(V)(-1)) = L_x^{-1}$.

1.6 Orbifold Riemann-Roch Theorem

At this point of the chapter it is worth mentioning another result, well known for smooth manifolds, which still holds true in the orbifold setting.

In particular, we will now present the analogue for orbifolds of the asymptotic Riemann-Roch Theorem for manifolds, which gives an asymptotic expansion for $h^0(X, L^k)$ as $k \to \infty$, where $X$ is a complex projective manifold and $L$ an ample line bundle over it. We will see that such an expansion is necessary in order to define the Donaldson-Futaki invariant of test configurations of polarised orbifolds, which is a key tool in the notion
of K-stability.

The asymptotic expansion we present can be obtained as a consequence of two main theorems: Kawazaki’s orbifold Riemann-Roch theorem, for a proof of which we refer to [15], and Toen’s for Deligne-Mumford stacks, [36]. We do not give the proofs of these results for two main reasons: firstly we are more interested in the structure of the expansion, which will be discussed later, rather than in the computations needed to obtain the coefficients of such an expansion; secondly, the arguments require the use of integration theory of Chern-Weil forms on orbifolds and intersection theory on Deligne-Mumford stacks, a treatment of which is beyond the scope of this thesis.

We thus have the following

**Proposition 1.3.** Let \((X, L)\) be a polarised orbifold of dimension \(n\). Then, the following formula holds true:

\[
h^0(L^k) = \frac{\int_X c_1(L)^n}{n!} k^n - \frac{\int_X c_1(L)^{n-1}c_1(K_{orb})}{2(n-1)!} k^{n-1} + \tilde{o}(k^{n-1}),
\]

\(\tilde{o} + \text{ as } k \to \infty,\) where here and in what follows we define \(\tilde{o}(k^{n-1})\) to be a function of \(k\) that can be written as \(r(k)\delta(k) + O(k^{n-2})\), where \(r(k)\) is a polynomial of degree \(n - 1\) and \(\delta(k)\) is periodic in \(k\) with period \(m = \text{ord}(X)\) and average \(0:\)

\[
\delta(k) = \delta(k + m), \quad \sum_{j=1}^m \delta(u) = 0.
\]

We now study a simple example, which shows very well why and how periodic terms come up in the orbifold setting. As one might guess, they are due to non-trivial cyclic stabiliser groups.

**Example 1.16.** Consider the orbifold Riemann Surface \(X = \mathbb{W}P^1(1, m)\). As discussed in a previous section, this is an orbifold and a straightforward calculation shows that it has only one singular point at \([1,0]\) with stabiliser group equal to \(\Gamma = \mathbb{Z}/m\mathbb{Z}\). We polarise \(X\) with \(L = O_X(1)\) and we want to compute \(h^0(L^k)\) for \(k \gg 0\). One has just to note that the section of \(L^k\) are just the \(\Gamma\)-invariant sections of \(O_{\mathbb{W}P^1}(k)\), a basis of which is indeed given by \(x^k, x^{k-m}y^m, \ldots, x^{k-m\lfloor \frac{k}{m} \rfloor}y^{m\lfloor \frac{k}{m} \rfloor}\). Thus, it follows that \(h^0(L^k) = \lfloor \frac{k}{m} \rfloor + 1\).

Observe now that, if we define \(\delta(k) = \lfloor \frac{k}{m} \rfloor - \frac{k}{m} + \frac{m-1}{2m}\), then \(\delta\) satisfies the conditions of Proposition 1.3, which means that

\[
h^0(L^k) = \frac{k}{m} + 1 - \frac{m-1}{2m} + \delta(k)
\]

is exactly the expansion in (1.2).
Chapter 2

K-stability and cscK metrics

This chapter will be devoted to the study of the second main topic of the present work, which is the notion of K-stability, introduced by S. K. Donaldson. Over the last years, this notion has been very fruitfully applied to the study of Kähler metrics with constant scalar curvature on compact complex manifolds. The interest in the relations between such metrics and K-stability has been so huge that it has led to many important results in both algebraic and differential geometry, which have showed how these two different areas of Geometry can merge to give rise to a beautiful mathematical theory.

The chapter is structured as follows: we will state the cscK problem, which is the problem of finding a Kähler metric with constant scalar curvature on a compact complex manifold in a prescribed cohomology class. Then we will see that it is not always possible to find such metrics, due to obstructions of both algebraic and differential geometric nature. In particular, we will be interested in the algebro-geometric notion of K-(semi)stability, the definition of which will occupy most of the chapter. We will give definitions and theorems both in the manifold and orbifold settings, since this second case is built up in complete analogy with the first one.

2.1 CscK metrics

Let $X$ be a compact connected Kähler manifold of (complex) dimension $n$. Fix a Kähler cohomology class $\Omega \in H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{R})$ and a Kähler metric $g$ such that its associated form $\omega_g$ lies in $\Omega$. Then, one can consider the scalar curvature of $g$ which, by definition, is $S(\omega) = \text{Tr} \omega_g \text{Ric}(\omega_g)$. Note that $S(g) \in C^\infty(X)$ and that, in local coordinates, it can be expressed as

$$S(\omega) = -g^{ij} \partial_i \partial_j \log \det(g_{k\bar{l}}).$$

We can average $S(\omega)$ and consider the number

$$\hat{S} = \int_X S(\omega) \frac{\omega^n}{n!}.$$
and it turns out that this is a fixed topological quantity: indeed a simple computation leads us to write
\[ \hat{S} = \frac{n \int_X c_1(X) \cup \Omega^{n-1}}{\int_{\Omega^n}}, \]
where \( c_1(X) = c_1(TX) \) and \( TX \) is the holomorphic tangent bundle of \( X \). This equality says that, once we fix the Kähler class \( \Omega \), \( \hat{S} = \hat{S}(\omega) \) does not depend on the choice of \( \omega \in \Omega \). Thus, it is natural to ask if there exists a Kähler metric \( g \), whose associated form \( \omega \) lies in \( \Omega \), such that
\[ S(\omega) = \hat{S}. \] (2.1)
We will call such metric a constant scalar curvature Kähler metric (cscK, for brevity) and (2.1) is called the cscK equation.

Example 2.1. A fundamental example of cscK metrics is given by Kähler-Einstein metrics, which are Kähler metrics such that
\[ \text{Ric}(\omega) = \lambda \omega, \]
where, up to rescaling, \( \lambda \) can be taken equal to 1, \(-1\) or 0. Indeed, if for example \( \omega \) satisfies \( \text{Ric}(\omega) = \omega \), we have
\[ S(\omega) = \text{Tr}_{\omega} \text{Ric}(\omega) = \text{Tr}_{\omega} \omega = n. \]
Conversely, it can be shown that if \( \omega \) is a cscK metric and \( c_1(X) = [(2\pi)^{-1} \text{Ric}(\omega)] = \lambda [\omega] \), then \( \omega \) is Kähler-Einstein.

Establishing necessary and sufficient conditions for the solvability of the cscK equation when \( n > 1 \) has been one of the main problems in Kähler geometry of the last forty years, since the work of E. Calabi in the 1970's.
There are indeed some effective obstructions to the solvability of the cscK equation (2.1); in this chapter we will develop the theory of K-stability for manifolds and orbifolds, which is one example of such obstructions. Moreover, note that, in our case, the cohomology class in which we want to find a cscK metric is just the first Chern class \( c_1(L) \) of the (orbi-)ample line bundle \( L \) of the pair \( (X, L) \).

Remark 2.1. It is important to note that all the quantities mentioned above make sense even in the orbifold setting and finding a solution to equation 2.1, where \( \omega \) is an orbifold Kähler form, is still an interesting problem, with applications in the theory of Sasaki-Einstein metrics, see for example [31].

2.2 Test configurations

This section is concerned with test configurations, which are the main tool used to define the notion of K-stability, both in the manifold and orbifold settings. Roughly speaking, a test configuration for a polarised (V-)manifold \((X, L)\) is a one parameter deformation of such pair and one is interested in the geometry of the limit for \( C^* \ni t \to 0 \) of
this deformation.

Let $X$ be a connected smooth complex projective variety of complex dimension $n$ and $L \to X$ an ample line bundle.

**Definition 2.1 (Test configuration).** A test configuration $(\mathcal{X}, \mathcal{L})$ for $(X, L)$ is a flat family $\mathcal{X} \to \mathbb{C}$ together with a line bundle $\mathcal{L}$ on the total space $\mathcal{X}$ such that

- the line bundle $\mathcal{L}$ is relatively ample;
- the total space $\mathcal{X}$ is endowed with an action of $\mathbb{C}^*$ which covers the natural action of $\mathbb{C}^*$ on $\mathbb{C}$;
- the $\mathbb{C}^*$-action of $\mathcal{X}$ admits a natural lift to $\mathcal{L}$, i.e. it is $\mathcal{L}$-linearised;
- the fibre $(\mathcal{X}_t, \mathcal{L}_t)$ is isomorphic as a polarised manifold to $(X, L')$ for some positive integer $r$ (it follows that the general fibre $(\mathcal{X}_{t \in \mathbb{C}^*}, \mathcal{L}_{t \in \mathbb{C}^*})$ is isomorphic to $(X, L')$).

The integer $r$ is called exponent of the test configuration.

**Remark 2.2.** Since the action of $\mathbb{C}^*$ on $\mathcal{X}$ covers the natural action $t \cdot z = tz \in \mathbb{C}$ and $0 \in \mathbb{C}$ is fixed by this action, there is a natural action of $\mathbb{C}^*$ on the central fibre $\mathcal{X}_0$. Furthermore, while the general fibre $\mathcal{X}_t, t \neq 0$ is a smooth subscheme of $\mathcal{X}$, the central fibre $\mathcal{X}_0$ may in general be singular, reducible and non-reduced.

The definition of orbifold test configuration is very similar to Definition 2.1: it is a one-parameter deformation of a polarised orbifold $(X, L)$. The only difference concerns the structure of the total space $\mathcal{X}$; indeed, if in the manifold case $\mathcal{X}$ is a scheme, in the orbifold setting it is an orbi-scheme, which is a geometric structure that generalises the notion of scheme just as (smooth) orbifolds extend the notion of manifold. Since we deal with orbifolds that can be embedded in weighted projective space by the Orbifold Kodaira Embedding Theorem, we can restrict our attention to a special class of orbi-schemes, i.e. those that can be obtained with the following construction.

**Example 2.2 (OrbiProj construction).** Given a finitely generated graded ring $R = \oplus_{k \geq 0} R_k$, we can form the scheme $\text{Proj} R$ in the usual way \([13]\). The orbi-scheme structure on $\text{Proj} R$ is given by describing the orbi-scheme charts: fix a homogeneous element $r \in R_+$ and consider the Zariski open subset $\text{Spec} R_{(r)}$, where $R_{(r)}$ is the degree zero part of the localised ring $r^{-1} R$. Then,

$$\text{Spec} \frac{R}{(r-1)} \to \text{Spec} R_{(r)}$$

is the orbichart, $R/(r-1)$ is the quotient of $R$ by the ideal $(r-1)$. The map to $R_{(r)}$ sets $r$ to 1. Note that we can express the orbifold $\text{Proj}$ of the graded $R$ as the quotient of $\text{Spec}(R) \setminus \{0\}$ by the action of $\mathbb{C}^*$ induced by the grading.
Example 2.3. Just as in Example 1.13, we can construct orbi-schemes with cyclic quotient singularities in codimension one as log pairs of the form \((X, D)\), where \(X\) is a scheme and \(D\) a Cartier divisor of \(X\). To this purpose, consider a projective scheme \((X, L)\), i.e. a scheme \(X\) and an ample line bundle \(L\) on \(X\), and a Cartier divisor \(D \subset X\). From these data we can construct the so called Cadman’s \(r\)-th root orbi-scheme \((X, (1 - \frac{1}{r})D)\), [6]. The underlying space of such orbi-scheme is the scheme \(X\) but with stabiliser \(\mathbb{Z}/m\mathbb{Z}\) along \(D\) and, in the above notation, it is simply
\[
\left(X, \left(1 - \frac{1}{r}\right)D\right) = \text{Proj} \bigoplus_{k \geq 0} H^0 \left(X, O \left(\left\lfloor \frac{k}{r} \right\rfloor \right) \otimes L^k \right).
\] (2.2)

We can now proceed giving the definition of orbifold test configuration. Thus, fix a compact \(n\)-dimensional polarised orbifold with cyclic quotient singularities \((X, L)\).

Definition 2.2 (Orbifold test configuration). A test configuration for \((X, L)\) consists of a pair \((\pi : \mathcal{X} \to \mathbb{C}, \mathcal{L})\) where \(\mathcal{X}\) is an orbi-scheme, \(\pi\) is flat and \(\mathcal{L}\) is an ample orbi-line bundle along with a \(\mathbb{C}^*\)-action such that
\begin{itemize}
  \item the action is linear and covers the usual action on \(\mathbb{C}\);
  \item the general fibre \(\pi^{-1}(t)\) of the test configuration is \((X, L)\).
\end{itemize}

Remark 2.3. Note that, just as in the manifold case, the central fibre \(\mathcal{X}_0 = \pi^{-1}(0)\) will not itself be an orbifold, as it may have scheme structure or entire components consisting of points with non-trivial stabilisers.

Remark 2.4. Recall from Section 1.3 of the previous chapter that a polarised orbifold can be embedded into a weighted projective space, \(X \hookrightarrow \mathbb{W}P\), where \(\mathbb{W}P = \mathbb{W}P(\oplus_{k \geq 0} H^0(X, (L^*)^{k+i}))\) such that the pullback of \(O_{\mathbb{W}P}(1)\) is \(L\). Now take a 1-parameter subgroup of the automorphisms of \(\mathbb{W}P\), i.e. a monomorphism \(\mathbb{C}^* \hookrightarrow \text{Aut}(\mathbb{W}P)\) and consider the family \(\{(\mathcal{X}_t, \mathcal{L}_t) = t.(X, L)\}_{t \in \mathbb{C}^*}\), which naturally becomes a test configuration by adding the flat limit \(\mathcal{X}_0 = \lim_{t \to 0} \mathcal{X}_t\). Conversely, every test configuration gives rise to a 1-parameter subgroup of \(\text{Aut}(\mathbb{W}P)\). Indeed, the subgroup can be recovered from the test configuration via the induced \(\mathbb{C}^*\)-action on the dual of the vector space \((\pi_*, \mathcal{L})_0\). It can be shown that these two operations are mutual inverses, see for example [21]. This proves that test configurations for a polarised orbifold \((X, L)\) are in one-to-one correspondence with 1-parameter subgroups of the automorphism group of the weighted projective space in which \(X\) is embedded via sections of \(L\). Obviously, this is true even for manifolds, since, in this setting, they constitute a particular case (manifolds are just orbifolds with trivial stabiliser group at each point).

2.3 Donaldson-Futaki invariants and K-stability

In this section we will define the Donaldson-Futaki invariant for (orbifold) test configurations and give the notion of (orbifold) K-stability.
Let us start by considering a polarised manifold \((X, L)\). Given a test configuration \((\mathcal{X}, \mathcal{L})\), we are concerned with asymptotic expansions of the dimension of the space of sections of the invertible sheaf \(\mathcal{L}_0 \rightarrow \mathcal{X}_0\) and the trace of the induced \(\mathbb{C}^*\)-action on such a space \(\text{Tr} H^0(\mathcal{X}_0, \mathcal{L}_0^k)\).

**Remark 2.5.** It is worth explaining what we mean by trace (or total weight) of a \(\mathbb{C}^*\)-action on a finite dimensional complex vector space \(V\) of dimension \(n\); such action is given by a homomorphism \(\alpha : \mathbb{C}^* \rightarrow \text{GL}(V)\) and, then, it can be shown (see, for example, [18]) that there is a basis \(\{v_1, \ldots, v_n\}\) of \(V\) and integers \(\lambda_1, \ldots, \lambda_n\) such that the action is represented as follows:

\[
\mathbb{C}^* \ni t \mapsto \begin{bmatrix}
 t^{\lambda_1} & 0 & \cdots & 0 \\
 0 & t^{\lambda_2} & \cdots & 0 \\
 \vdots & \ddots & \ddots & \vdots \\
 0 & \cdots & 0 & t^{\lambda_n}
\end{bmatrix}.
\]

We define the trace of the action as the integer \(\text{Tr}(V) := \sum_{i=1}^{n} \lambda_i\) and \(\lambda_i\) is called the \(i\)-th weight of the action. Actually, \(\text{Tr}(V)\) can be computed as follows: let \(A\) be the infinitesimal generator of the action \(\alpha\), i.e. an endomorphism \(A : V \rightarrow V\) such that \(\mathbb{C}^*\) acts on \(V\) as \(\mathbb{C}^* \ni t \mapsto e^{tA} : V \rightarrow V\); then it is obvious that \(\text{Tr}(V)\) is the trace of the linear map \(A\).

In order to expand \(h^0(\mathcal{L}_0^k)\) we use the Hirzebruch-Riemann-Roch Theorem. More precisely, we use the following consequence, [14].

**Theorem 2.1.** Let \(X\) be a compact complex manifold and \(L \rightarrow X\) an ample line bundle over \(X\). Then, the following asymptotic expansion holds true for \(k \rightarrow \infty\):

\[
h^0(X, L^k) = k^n \int_X c_1(L)^n n! + k^{n-1} \int_X c_1(L)^{n-1} c_1(X) 2(n-1)! + O(k^{n-2}). \tag{2.3}
\]

Since, by flatness, we have \(h^0(\mathcal{X}_0, \mathcal{L}_0^k) = h^0(\mathcal{X}_1, \mathcal{L}_1^k)\), applying the previous result, we see that, for \(k \gg 0\), there are \(a_0, a_1 \in \mathbb{Z}\) such that

\[
h^0(\mathcal{X}_0, \mathcal{L}_0^k) = a_0 k^n + a_1 k^{n-1} + O(k^{n-2}). \tag{2.4}
\]

Furthermore, an equivariant version of Theorem 2.1 enables us to write an analogue of the expansion (2.4) for the total weight of the \(\mathbb{C}^*\)-action. More explicitly, there are integers \(b_0, b_1 \in \mathbb{Z}\) such that, for \(k \gg 0\), we have

\[
\text{Tr}(H^0(\mathcal{X}_0, \mathcal{L}_0^k)) = b_0 k^{n+1} + b_1 k^n + O(k^{n-1}). \tag{2.5}
\]

In the above set-up, we can give the following definition.

**Definition 2.3.** The **Donaldson-Futaki invariant** of a test configuration \((\mathcal{X}, \mathcal{L})\) is the rational number

\[
F(\mathcal{X}) = \frac{a_1 b_0 - a_0 b_1}{a_0^2},
\]

where \(a_0, a_1\) and \(b_0, b_1\) are the coefficients of expansions (2.4) and (2.5) respectively.
2.3 Donaldson-Futaki invariants and K-stability

Now we aim at extending the definition of the Donaldson-Futaki invariant to orbifold test configurations. To this purpose, let us consider an orbifold test configuration \((X, L)\) for a polarised orbifold \((X, L)\). In complete analogy with the manifold case, we know that, by Proposition 1.3 of the previous chapter, there exist rational numbers \(a_0, a_1\) such that, as \(k \to \infty\),

\[
h^0(X, L^k) = a_0 k^n + a_1 k^{n-1} + \hat{o}(k^{n-1}),
\]

where \(\hat{o}(k^{n-1}) = r(k)\delta(k) + O(k^{n-2})\), \(r(k)\) is a polynomial of degree \(n - 1\) and \(\delta(k)\) is a periodic function in \(k\) of period \(m = \text{ord}(X)\) and average 0.

Moreover, in [20], Proposition 1.9, Ross and Thomas prove that an analogous expansion holds true for the total weight \(\text{Tr}(H^0(X_0, L_0^k))\):

\[
\text{Tr}(H^0(X_0, L_0^k)) = b_0 k^{n+1} + b_1 k^n + \hat{o}(k^n).
\]

We see that expansions (2.6) and (2.7) are very similar to (2.4) and (2.5). Indeed, the only difference is the periodic term \(\delta\), which is due to non-trivial cyclic stabilisers.

Starting from (2.6) and (2.7), we can give the definition of Orbifold Donaldson-Futaki invariant of an orbifold test configuration exactly in the same way of Definition 2.3. For sake of completeness, we write down the following

**Definition 2.4.** The orbifold Donaldson-Futaki invariant of an orbifold test configuration \((X, L)\) for a polarised orbifold is the rational number

\[
F(X) = \frac{b_0 a_1 - a_0 b_1}{a_0^2},
\]

where \(a_0, a_1\) and \(b_0, b_1\) are the coefficients of expansions (2.6) and (2.7) respectively.

**Remark 2.6.** Note that, both in the manifold and orbifold cases, the Donaldson-Futaki invariant can be obtained directly from an asymptotic expansion. Indeed, if \((X, L)\) is an orbifold test configuration for a polarised orbifold \((X, L)\) and \(F(X)\) denotes its Donaldson-Futaki invariant, then

\[
\frac{h^0(X, L_0^k)}{k \text{Tr } H^0(X_0, L_0^k)} = C + \frac{F(X)}{k} + O\left(\frac{1}{k^2}\right), \quad k \to \infty.
\]

Furthermore, this identity shows that \(F(X)\) is invariant under taking tensors powers of the (orbi-)line bundle \(L\).

In order to give the definition of K-(semi)stable manifold (or orbifold) we need some additional notions.

Given a (orbifold) test configuration \((X, L)\), we can define the \(L^2\) norm \(\|(X, L)\|\) of \((X, L)\), which is a non-negative real number computed as follows: denote with \(A_k\) the endomorphism that generates the \(C^*\)-action on the space \(H^0(X_0, L_0^k)\); as for \(\text{Tr } A_k = \text{Tr } H^0(X_0, L_0^k)\), we can expand \(\text{Tr } A_k^2\) as

\[
\text{Tr } A_k^2 = d_0 k^{n+2} + O(k^{n+1}).
\]
We define $||X||^2 = d_0 - b_0^2/a_0$, where $a_0$ and $b_0$ are the coefficients of expansions (2.4) and (2.5) respectively. There is also a corresponding notion of formal $L^2$ inner product (see [32]).

We proceed now giving the definitions of trivial and product test configuration.

**Definition 2.5.** A test configuration $(\mathcal{X}, \mathcal{L})$ for a polarised manifold $(X, L)$ is trivial if it is equivariantly isomorphic to the test configuration $(X \times \mathbb{C}, L)$ with trivial action on $X$, outside a closed subscheme of codimension $\geq 2$.

**Definition 2.6.** A test configuration $(\mathcal{X}, \mathcal{L})$ is a product if it is equivariantly isomorphic to the test configuration $(X \times \mathbb{C}, L)$ with a not necessarily trivial action on $X$, outside a closed subscheme of codimension $\geq 2$.

**Definition 2.7.** A polarised manifold $(X, L)$ is K-semistable if for all test configurations $(\mathcal{X}, \mathcal{L})$ the Donaldson-Futaki invariant is non-negative. It is K-stable if the strict inequality $F(\mathcal{X}, \mathcal{L}) > 0$ holds for all non-trivial test configurations.

**Remark 2.7.** It can be proved that if a polarised manifold is K-stable, then the automorphism group $\text{Aut}(X, L)$ is a discrete group.

The following definition is the one used in the celebrated theorem of Chen, Donaldson and Sun (Theorem 2.6), which gives a final answer to the problem of finding solutions to the Kähler-Einstein equation in the Fano case.

**Definition 2.8.** A polarised manifold $(X, L)$ is K-polystable if $F(\mathcal{X}, \mathcal{L}) > 0$ except for product test configurations $(\mathcal{X}, \mathcal{L})$, where $F(\mathcal{X}, \mathcal{L}) = 0$.

**Remark 2.8.** We say that $(X, L)$ is K-unstable if $F(\mathcal{X}, \mathcal{L}) < 0$ for some test configuration $(\mathcal{X}, \mathcal{L})$. We say that $(X, L)$ is strictly K-semistable if there exists a non-trivial, non-product test configuration $(\mathcal{X}, \mathcal{L})$ such that $F(\mathcal{X}, \mathcal{L}) = 0$.

A theorem by Donaldson - which will be obtained as a corollary of another result in the next section - links the notion of K-stability to the solvability of the cscK equation.

**Theorem 2.2.** If a polarised manifold $(X, L)$ is cscK, i.e. the class $c_1(L)$ contains a cscK metric, then it is K-semistable.

The previous result has been partially strengthened by Stoppa in [26] and [28].

**Theorem 2.3.** Let $(X, L)$ be a polarised manifold that admits a cscK metric in $c_1(L)$ and such that the automorphism group $\text{Aut}(X, L)$ is discrete. Then $(X, L)$ is K-stable.

**Corollary 2.1 ([28]).** A polarised manifold is K-stable if and only if $F(\mathcal{X}, \mathcal{L}) > 0$ for all normal test configurations which are not isomorphic to $(X \times \mathbb{C}, L)$ with trivial action on $X$.

**Remark 2.9.** The more obvious definition of triviality for test configurations, for which a test configuration $(\mathcal{X}, \mathcal{L})$ is trivial if it is equivariantly isomorphic to $(X \times \mathbb{C}, L)$ with trivial action on $X$, is wrong, essentially for dimensional reasons. This was first pointed out by Li and Xu, [17]. See also [28].
There is an alternative definition of K-stability, due to Székelyhidi [30], which has attracted much attention recently.

**Definition 2.9** (K-stability in the $L^2$ sense). A polarised manifold $(X, L)$ is **K-stable in the $L^2$ sense** if $F(\mathcal{H}^\prime, \mathcal{L}) > 0$ whenever $\| (\mathcal{H}^\prime, \mathcal{L}) \|_{L^2} > 0$, where $\| \cdot \|_{L^2}$ denotes the $L^2$ norm of test configurations defined above.

A result by Dervan [9] says that the previous definition is equivalent to the original one.

**Theorem 2.4** (Dervan, [9]). A polarised manifold is K-stable if and only if it is K-stable in the $L^2$ sense.

Motivated by this, we might contemplate an alternative definition of polystability too. To this purpose, fix a polarised manifold $(X, L)$ and pick a maximal complex torus $\mathbb{T}^r \subset \text{Aut}(X, L)$ and choose generators $\xi_1, \ldots, \xi_r$. Then, for every test configuration $(\mathcal{H}^\prime, \mathcal{L})$ set $\| \pi_{\mathcal{T}^r}(\mathcal{H}^\prime, \mathcal{L}) \|_{L^2}^2 := \sum_{i=1}^r \left( \frac{\langle \mathcal{H}^\prime, \xi_i \rangle_{L^2}}{\| \xi_i \|_{L^2}^2} \right)^2$, where $\langle \cdot, \cdot \rangle_{L^2}$ denotes the formal $L^2$ product on test configurations (see [32]). Note that $\xi_i$ indicates both the generator in $\text{Aut}(X, L)$ and the test configuration generated by such one-parameter subgroup of $\text{Aut}(X, L)$.

**Definition 2.10.** We say that a polarised manifold $(X, L)$ is **K-polystable in the $L^2$ sense** if $F(\mathcal{H}^\prime, \mathcal{L}) > 0$ whenever $\| (\mathcal{H}^\prime, \mathcal{L}) \|_{L^2} > \| \pi_{\mathcal{T}^r}(\mathcal{H}^\prime, \mathcal{L}) \|_{L^2}$.

**Remark 2.10.** It is still a work in progress to prove if this definition is equivalent to the original one.

**Remark 2.11.** Note that all of the previous definitions are immediately extended to orbifolds.

### 2.4 Main results

In this section we will recollect the most important results proved in the last years that link the notion of K-stability together with the resolvability of the cscK equation.

Firstly, we state a famous conjecture made by S. T. Yau, G. Tian and S. Donaldson concerned with a characterisation of the existence of cscK metrics on polarised manifolds.

Note that, from now on, we will say that a polarised manifold $(X, L)$ is cscK if it admits a cscK metric $\omega \in c_1(L)$.

**Conjecture 2.1** (YTD Conjecture). A polarised manifold is cscK if and only if it is K-polystable.

In the special case when $L = K_X^{\pm 1}$, where $K_X$ is the canonical bundle of $X$, the cscK equation is equivalent to the Kähler-Einstein equation

$$\text{Ric}(\omega) = \mp \omega,$$

as explained in Example 2.1. In these cases the problem of finding solutions to the cscK equation has been solved in full generality by the following theorems.
2.4 Main results

**Theorem 2.5** (Aubin, Yau, [29]). Let \( X \) be a compact Kähler manifold of dimension \( n \) such that \( c_1(X) < 0 \). Then, there exists a unique Kähler form \( \omega \in -2\pi c_1(X) \) such that
\[
\text{Ric}(\omega) = -\omega.
\]

**Theorem 2.6** (Chen, Donaldson, Sun, [7]). Let \( X \) be a compact Kähler manifold such that \( c_1(X) > 0 \). Then, the Kähler-Einstein equation
\[
\text{Ric}(\omega) = \omega, \quad \omega \in 2\pi c_1(X),
\]
has a solution if and only if \( (X, K_X^{-1}) \) is polystable. Moreover, if the solution exists, it is unique.

A fundamental result in the general theory of K-stability for polarised manifolds, i.e. without making further assumptions on the ample line bundle \( L \), that relates this notion to the solvability of the cscK equation, is Donaldson’s lower bound to the Calabi functional.

**Definition 2.11.** Let \( X \) be a compact Kähler manifold and let \( \mathcal{K}(X) \) denote the cone of Kähler forms on \( X \). The Calabi functional is the map
\[
\text{Cal} : \mathcal{K}(X) \to \mathbb{C}, \quad \text{Cal}(\omega) = \int_X \left(S(\omega) - \hat{S}_\omega\right)^2 \omega^n,
\]
where \( S(\omega) \) and \( \hat{S}_\omega \) denote the scalar curvature and the average of the scalar curvature of \( \omega \) respectively.

**Theorem 2.7** (Donaldson, [10]). For a polarised manifold \( (X, L) \)
\[
\inf_{\omega \in c_1(L)} \text{Cal}(\omega) \geq -\sup \frac{F(\mathcal{X}', \mathcal{L})}{|||\mathcal{X}'\mathcal{L}|||},
\]
where the supremum is taken over all test configurations \( (\mathcal{X}', \mathcal{L}) \) for \( (X, L) \).

Donaldson’s estimate has an immediate and important consequence, which is the K-semistability result we mentioned in the previous section.

**Corollary 2.2.** If \( c_1(L) \) admits a cscK representative, \( (X, L) \) is K-semistable.

As already discussed in the previous section, one would like to strengthen the previous result, proving that if a polarised manifold is cscK, then it is K-stable. As has been said, this was partially done by Stoppa in [25] and [26], under certain hypotheses on the automorphism group of \( (X, L) \). Indeed, in these papers the following result is proved.

**Theorem 2.8** (Stoppa, [26]). Let \( (X, L) \) be a polarised manifold that admits a cscK metric in \( c_1(L) \) and such that the automorphism group \( \text{Aut}(X, L) \) is discrete. Then \( (X, L) \) is K-stable.

The proof of the previous theorem is based on a deep analytical result by Arezzo and Pacard [2], who could prove that the problem of lifting a cscK metric from a compact Kähler manifold to its blow-up at a finite set of points is unobstructed. More precisely, we have the following
2.4 Main results

Theorem 2.9 (Arezzo, Pacard, [2]). Let \((X, \omega)\) be a constant scalar curvature Kähler manifold or Kähler orbifold of dimension \(m\) with isolated singularities. Assume that there is no non-zero holomorphic vector field vanishing somewhere on \(X\), i.e. assume that the automorphism group of \(X\) is discrete. Then, given finitely many smooth points \(p_1, \ldots, p_n \in X\) and positive numbers \(a_1, \ldots, a_n > 0\), there is \(\varepsilon_0 > 0\) such that \(Bl_{p_1, \ldots, p_n} X\) carries constant scalar curvature Kähler forms

\[
\omega_{\varepsilon} \in \pi^*\omega - \varepsilon^2(a_1 PD[E_1] + \cdots + a_n PD[E_n]),
\]

where \(PD[E_i]\) is the Poincaré dual of the \((2m - 2)\)-homology class of the exceptional divisor of the blow-up at \(p_j, j = 1, \ldots, n\), and \(\varepsilon \in (0, \varepsilon_0)\).

The argument for the proof of Stoppa’s result is called “blow-up method”: by contradiction, suppose that there is a non-trivial test configuration \((\mathcal{X}, \mathcal{Z})\) for the polarised cscK manifold \((X, L)\) such that its Donaldson-Futaki invariant vanishes, \(F(\mathcal{X}) = 0\). Then, as shown in the next chapter, one can construct a non-trivial test configuration \((\hat{\mathcal{X}}, \hat{\mathcal{Z}})\) for the blow-up \(Bl_q X\), polarised by \(\pi^* L^\gamma \otimes O(-E)\), which is ample for \(\gamma \gg 0\), where \(\pi: Bl_q X \to X\) is the blow-up map with exceptional divisor \(E\). More importantly, in \([25]\), the author proves a formula which relates \(F(\mathcal{X})\) with \(F(\hat{\mathcal{X}})\), as the parameter \(\gamma \to \infty\):

\[
F(\hat{\mathcal{X}}) = F(\mathcal{X}) - C(q)\gamma^{1-n} + O(\gamma^{-n}). \tag{2.8}
\]

We will discuss in the next chapter the nature of the constant \(C(q)\), which crucially depends on the point \(q \in X\). Indeed, it is possible to prove that \(q\) can be chosen such that \(C(q) > 0\) and we claim that, with such choice of the point \(q\), we can conclude the proof of Theorem 2.8: in fact, by assuming that \(F(\mathcal{X}) = 0\) and taking the parameter \(\gamma\) sufficiently large, (2.8) implies that \(F(\hat{\mathcal{X}}) < 0\); thus, \((Bl_q X, \pi^* L^\gamma \otimes O(-E))\) is K-unstable. But, by the Arezzo-Pacard Theorem we know that, for \(\gamma \gg 0\), \((Bl_q X, \pi^* L^\gamma \otimes O(-E))\) is cscK, and, thus, K-semistable and we get a contradiction.

In the orbifold setting, K-stability is still an obstruction to the existence of orbifold cscK metrics. Indeed Donaldson’s Theorem and, more importantly, one of its consequences, namely Corollary 2.2, have an analogue in the context of orbifold K-stability.

The general set-up is that of Section 1.5 of the first chapter: consider a polarised orbifold \((X, L)\). By Theorem 1.1, for \(k \gg 0\), \(X\) can be embedded in the weighted projective space \(\mathbb{W}P(\oplus_i H^0(L^{k+i}))\). By ampleness of \(L\), we can suppose that \(L\) has an hermitian metric \(h\) with positive curvature \(2\pi a\) - this fact is recalled in Remark 1.12.

Now, let \((\mathcal{X}, \mathcal{Z})\) be a non-trivial orbifold test configuration for \((X, L)\) embedded in \(\mathbb{W}P(V) \times C\) (with \(V = \oplus_i H^0(\mathcal{X}_0, \mathcal{Z}_0^{k+i})\)), induced by a \(C^*\)-action on \(\mathbb{W}P(V)\) that takes \(X\) to \(\mathcal{X}_0\). The key estimate, which is the analogue of Donaldson’s lower bound to the Calabi functional, is given by the following result.

Proposition 2.1 (Ross, Thomas, [20]). In the set-up above, the following estimate holds true for \(k \gg 0\):

\[
k^{n+2} \int_X (S(\omega) - \tilde{S}_\omega)^2 \omega^n + O(k^{n+1}) \geq Ck^2 (-k^{-1}F(\mathcal{X}) + O(k^{-2})), \tag{2.9}
\]
for some positive constant $C > 0$.

Just as Theorem 2.7, the previous proposition has the following important consequence, which is a perfect analogue of Corollary 2.2.

**Corollary 2.3.** Let $(X, L)$ be a polarised orbifold with cyclic stabilisers. If $X$ admits an orbifold Kähler metric $\omega \in K(c_1(L))$, with constant scalar curvature then $(X, L)$ is K-semistable.

The approach for the proof of Proposition 2.1 follows that one given by Donaldson in [11]. Then, it is natural to ask if Theorem 2.8 too can be extended to orbifolds. A proof of such an extension is far from being immediate. Indeed, in order to use an orbifold analogue of the blow-up method, we need the following results:

1. given an orbifold test configuration $(\mathcal{X}, \mathcal{L})$ for a polarised orbifold $(X, L)$ it is possible to construct another orbifold test configuration $(\hat{\mathcal{X}}, \hat{\mathcal{L}})$ for $\text{Bl}_q X$, where $q \in X$, provided that we have a good definition of blow-up at a point or, more in general, at a sub-orbischeme, in the orbifold category;

2. once the orbifold test configuration $(\hat{\mathcal{X}}, \hat{\mathcal{L}})$ is constructed, there exists a formula that relates $F(\hat{\mathcal{X}})$ and $F(\mathcal{X})$ in the same way of (2.8);

3. there is an analogue of Arezzo-Pacard Theorem in the orbifold setting, i.e. the perturbation problem of lifting an orbifold cscK metric from $X$ to the blow-up $\text{Bl}_q X$ is unobstructed, provided that $\text{Aut}(X, L)$ is discrete.

It is straightforward to see that, by taking 1., 2. and 3. for granted, Theorem 2.8 can be immediately extended to orbifolds, since the arguments for the proof are exactly the same. In the next chapter we will present the construction 1. and give a partial answer to 2., i.e. we will prove a formula for special test configurations. On the other hand, it remains still an open problem to prove that we can choose the point $q \in X$ so that the constant $C(q)$ is strictly positive. Finally, we will make a precise statement of 3., inspired by a particular but relevant case, i.e. when the test configuration $(\mathcal{X}, \mathcal{L})$ is the deformation to the normal cone. We will present this result as a conjecture, for two main reasons: firstly, it is not known to the author if such a theorem has already been proved; secondly, its proof would require techniques from geometric analysis and PDEs, the study of which is beyond the purposes of this work.
Chapter 3

An extension of the Blow-up formula

We have two main purposes in this chapter. Firstly, we want to present an alternative proof of Stoppa’s blow-up formula for test configurations, which avoids the use of the asymptotic Riemann-Roch Theorem. Secondly, we want to show how this alternative approach can be extended to the context of orbifolds with cyclic quotient singularities in codimension one and used to prove an analogue of the aforementioned blow-up formula.

Even though our extension holds only under some smoothness assumptions, that will be made clear later in the chapter, it is useful to consider this particular case because it enables us to find out how the arguments for the proof have to be changed and generalised. It has to be noticed that most of the techniques used in both proofs are the same and this gives us the hope to completely extend the blow-up formula for manifolds in some future work.

The chapter is organised as follows: firstly, we recall how, given a test configuration for a polarised manifold \((X, L)\), we can construct a new test configuration for the pair \((\text{Bl}_p X, L^\gamma \otimes O(-E))\), where \(p \in X\) and \(E\) is the exceptional divisor of \(\text{Bl}_p X \to X\). We then show how to generalise this construction to the case of polarised orbifolds. Furthermore, we proceed to explain the alternative proof of the blow-up formula and, finally, we show the aforementioned extension and some of its possible applications. Our main techniques mainly involve the use of exact sequences of (orbi-)sheaves with no higher cohomology and of some combinatorial asymptotic expansions.

3.1 Blowing up test configurations

Let us consider a test configuration \((\mathcal{X}, \mathcal{L}) \to C\) for a polarised manifold \((X, L)\) and a (closed) point \(q \in \mathcal{X}_1\) in the fibre over \(1 \in C\), that can be identified with \(X\). The orbit \(C^* \cdot q\) is a locally closed subscheme in \(\mathcal{X}\) and let \((C^* \cdot q)^-\) be its schematic closure. Given
3.1 Blowing up test configurations

these data, it can be proved that, by blowing up the total space \( \mathcal{X} \) along the \( (C^\ast, q)^{-} \), one can construct a test configuration for the blow-up manifold \( \text{Bl}_q X \). More explicitly, we have the following

**Proposition 3.1.** Under the previous hypotheses and notations, for any \( \gamma \) sufficiently large, the family

\[
\hat{\mathcal{X}} = \text{Bl}_{(C^\ast, q)^{-}} \mathcal{X} \to \mathbb{C}
\]

together with the invertible sheaf

\[
\hat{\mathcal{L}} = \pi^* \mathcal{L}^\gamma \otimes \mathcal{O}((\pi^{-1}(C^\ast, q)^{-}))^{-1}
\]

is a test configuration for the polarised manifold

\[
(\text{Bl}_q X, \pi^* \mathcal{L}^\gamma \otimes \mathcal{O}((\pi^{-1}(q)))^{-1}).
\]

**Proof.** This is just a particular case of Lemma 3.2 in [27]. \( \square \)

**Remark 3.1.** Once the test configuration \( (\hat{\mathcal{X}}, \hat{\mathcal{L}}) \) is constructed, it is natural to ask what the central fibre \( (\hat{\mathcal{X}}_0, \hat{\mathcal{L}}_0) \) looks like. One might guess that the flat limit of blowups is the blow-up of the flat limit, i.e. that \( \hat{\mathcal{X}}_0 \) is isomorphic to \( \text{Bl}_{q_0} \mathcal{X}_0 \), where \( q_0 = \lim_{t \to 0} t \cdot q \). Indeed, if \( q_0 \) is a smooth point of the central fibre, it is easy to prove that the previous statement is true. On the other hand, without assuming the smoothness of the point \( q_0 \), there are counterexamples proving that, in the general case, \( \text{Bl}_{q_0} \mathcal{X}_0 \) is just a component of the central fibre \( \hat{\mathcal{X}}_0 \). One such counterexample, constructed by Donaldson, is described below.

**Example 3.1.** Consider the hypersurface \( M \) in \( \mathbb{C}^4 \) defined by the equation

\[
x_4 = x_1^2 + x_2^2 + x_3^2.
\]

Let \( C^\ast \) act on \( \mathbb{C}^4 \) by

\[
(x_1, x_2, x_3, x_4) \mapsto (t^{-a}x_1, t^{-a}x_2, t^{-a}x_3, t^{-b}x_4).
\]

If \( a, b > 0 \) then the origin is a (repulsive) fixed point for this action. Moreover, we get a family of hypersurfaces \( M_t \) defined by the equation

\[
x_4 = t^{2a-b}(x_1^2 + x_2^2 + x_3^2).
\]

Thus, if \( 2a - b > 0 \) the central fibre of this test configuration is the smooth subspace \( x_4 = 0 \). If \( 2a - b < 0 \) the limiting equation is \( x_1^2 + x_2^2 + x_3^2 = 0 \), which defines the subset \( \mathbb{C} \times \Gamma \subset \mathbb{C}^4 \), where \( \Gamma \) is a quadratic cone, so it is singular at the origin. Now, the blow-up of \( M_t \) at the origin can be regarded as a subset of \( \mathbb{C}^4 \times \mathbb{P}^3 \). So the total space of the blow-up test configuration is the closure in \( \mathbb{C}^4 \times \mathbb{P}^3 \times \mathbb{C} \) of a subset \( V \subset \mathbb{C}^4 \times \mathbb{P}^3 \times \mathbb{C}^\ast \).
3.2 Alternative proof of the blow-up formula

Consider a vector \((r_1, r_2, r_3, r_4)\) such that \(r_1^2 + r_2^2 + r_3^2 \neq 0\). The line generated by this vector meets the hypersurface \(M_t\) at the point

\[
P = \frac{4}{r_1^2 + r_2^2 + r_3^2} (r_1, r_2, r_3, r_4).
\]

We can make this point as close to the origin as we like by taking \(t\) small. It follows that the closure of \(V\) contains the whole \(\mathbb{P}^3\) factor at \((0,0) \in \mathbb{C}^4 \times \mathbb{C}\). Thus the central fibre of the blow-up test configuration is given by \(\text{Bl}_0 M_0 \cup \mathbb{P}^3\).

3.2 Alternative proof of the blow-up formula

In this section we want to compute the Donaldson-Futaki invariant of the test configuration \((\tilde{X}, \tilde{L})\), in terms of the Donaldson-Futaki invariant of \((X, L)\), as the parameter \(\gamma \to \infty\).

The proof given in [26] exploits two main techniques: the asymptotic Riemann-Roch formula and the vanishing of the higher cohomology of a certain exact sequence of sheaves. We remind that, as seen in the last section of the first chapter, the first of these two results has an analogue in the orbifold setting. The main issue to face when dealing with the Orbifold Riemann-Roch Theorem is that, in order to get expansion (1.2), one has to compute integrals which can be very far from being easily solved. This computational obstruction forces us to give an alternative proof of the classical blow-up formula, which uses arguments that can be efficiently extended to the orbifold case. We will present a proof that holds under the following assumption: keeping the notation of the previous section, we suppose that \(q_0^*\) is a smooth point in the central fibre \(X_0^*\).

Obviously, this is a restrictive hypothesis, but still represents an interesting case.

According to the definition of Donaldson-Futaki invariant given in the previous chapter, we need to compute, for \(k \gg 0\):

- the dimension of the space of sections \(h^0(\tilde{X}_0^*, \tilde{L}_0^k)\);
- the trace of the induced action of \(\mathbb{C}^*\) on the space \(H^0(\tilde{L}_0^k)\).

In order to compute these quantities, we note that for \(k \gg 0\) there is a natural isomorphism

\[
H^0_{\text{Bl}_{q_0} X_0^*}(\pi^* \mathcal{L}_0^k \otimes O(\pi^{-1}(q_0))^{-k}) \cong H^0_{\mathcal{X}_0^*}(\mathcal{O}_0^* \mathcal{L}_0^k),
\]

induced by \(\pi_\ast O(\pi^{-1}(q_0))^{-k} \cong \mathcal{O}_0^*\) for \(k \gg 0\), where \(\mathcal{I}_0\) is the ideal sheaf at \(q_0\). Thus, since we have that \(\tilde{X}_0^* = \text{Bl}_{q_0} X_0^*\), we are interested in computing \(h^0_{\mathcal{X}_0^*}(\mathcal{O}_0^* \mathcal{L}_0^k)\) and \(\text{Tr}(H^0_{\mathcal{X}_0^*}(\mathcal{O}_0^* \mathcal{L}_0^k))\).

More precisely, we want to find the first two coefficients of their polynomial asymptotic expansions, i.e.

\[
h^0_{\mathcal{X}_0^*}(\mathcal{O}_0^* \mathcal{L}_0^k) = a_0 k^n + a_1 k^{n-1} + O(k^{n-2}),
\]

\[
\text{Tr}(H^0_{\mathcal{X}_0^*}(\mathcal{O}_0^* \mathcal{L}_0^k)) = b_0 k^n + b_1 k^{n-1} + O(k^{n-2}).
\]
Thus, let us consider the following exact sheaf sequence of $\mathbb{C}^*$-linearised sheaves on $\mathcal{X}_0$,

$$0 \to \mathcal{S}_0^k \mathcal{L}_0^k \to \mathcal{L}_0^k \to \mathcal{O}_{\mathcal{X}_0} \otimes \mathcal{L}_0^k (q_0) \to 0.$$ 

**Remark 3.2.** Here, $\mathcal{O}_{\mathcal{X}_0}$ is a skyscraper sheaf at $q_0$, with fibre the space of jets of holomorphic functions of order up to $k - 1$ near $q_0$. In other words, this is the space of polynomials in $n$ variables of degree up to $k - 1$. We see that our assumption of $q_0 \in \mathcal{X}_0$ being a smooth point is crucial in understanding the structure of the sheaf $\mathcal{O}_{\mathcal{X}_0}$.

We now use that, for $k \gg 0$, independently of $\gamma \geq \gamma_0$, the higher cohomology groups of this sequence vanish. We can write the following exact sequence of vector spaces:

$$0 \to H^0(\mathcal{X}_0, \mathcal{S}_0^k \mathcal{L}_0^k) \to H^0(\mathcal{L}_0^k) \to \mathcal{O}_{\mathcal{X}_0} \otimes \mathcal{L}_0^k (q_0) \to 0. \tag{3.1}$$

Thus, the following two identities

$$h^0(\mathcal{S}_0^k \mathcal{L}_0^k) = h^0(\mathcal{L}_0^k) - \dim \mathcal{O}_{\mathcal{X}_0} \otimes \mathcal{L}_0^k (q_0), \tag{3.2}$$

$$\text{Tr}(H^0(\mathcal{X}_0, \mathcal{S}_0^k \mathcal{L}_0^k)) = \text{Tr}(H^0(\mathcal{L}_0^k)) - \text{Tr}(\mathcal{O}_{\mathcal{X}_0} \otimes \mathcal{L}_0^k (q_0)) \tag{3.3}$$

hold true for $k \gg 0$.

**Lemma 3.1.** For $k \gg 0$, the following asymptotic expansion holds true:

$$\dim(\mathcal{O}_{\mathcal{X}_0} \otimes \mathcal{L}_0^k (q_0)) = \frac{k^n}{n!} + \frac{k^{n-1}}{2(n-2)!} + O(k^{n-2}).$$

**Proof.** Since $\mathcal{L}_0$ is a line bundle, we have that $\dim(\mathcal{O}_{\mathcal{X}_0} \otimes \mathcal{L}_0^k (q_0)) = \dim \mathcal{O}_{\mathcal{X}_0}$. But, by Remark 3.2, we have that $\dim \mathcal{O}_{\mathcal{X}_0} = \dim \mathbb{C}[z_1, \ldots, z_n]_{k-1} := \{ P \in \mathbb{C}[z_1, \ldots, z_n], \deg P \leq k - 1 \}$. By decomposing a generic polynomial in its homogeneous terms, it is immediate to see that

$$\dim \mathbb{C}[z_1, \ldots, z_n]_{k-1} = \sum_{i=0}^{k-1} \binom{n-1+i}{i} = \prod_{i=1}^{n} \frac{k-1+i}{i(n-i)!},$$

where the second identity can be easily proved by induction. Furthermore,

$$\prod_{i=1}^{n} \frac{k-1+i}{n!} = k(k+1) \cdots (k+n-1) = \frac{k^n + \frac{n(n-1)}{2}k^{n-1}}{n!} + O(k^{n-2}),$$

implying that

$$\dim \mathbb{C}[z_1, \ldots, z_n]_{k-1} = \frac{k^n}{n!} + \frac{k^{n-1}}{2(n-2)!} + O(k^{n-2}), \tag{3.4}$$

which concludes the proof. \qed
Lemma 3.2.
\[ h^0(\mathcal{L}_0^k) = h^0(\mathcal{L}_{\gamma,0}^k) - \frac{k^n}{n!} - \frac{k^{n-1}}{2(n-2)!} + O(k^{n-2}). \]

Now, as well as for the dimension, we use (3.3) to compute the trace of the \( C^* \)-action on the vector space \( H^0(\mathcal{X}_0, \mathcal{L}_0^k) \). Firstly, let us recall that, by definition, the weight of the action on the line \( \mathcal{L}_0(q_0) \) is the Hilbert-Mumford weight \( \mu_{\mathcal{L}_0}(q_0) \) for the induced action on \( \mathcal{X}_0 \) linearised by \( \mathcal{L}_0 \), see [33] for further details. Moreover, the simple following relation holds true for \( m > 0 \),

\[ \mu_{\mathcal{L}_0^m}(q_0) = m \mu_{\mathcal{L}_0}(q_0). \]

Thus, noting that the action on the space \( \mathcal{O}_{kq_0} \) does not depend on \( \gamma \), we can write

\[ \text{Tr}(\mathcal{O}_{kq_0} \otimes \mathcal{L}_0^k(q_0)) = \mu_{\mathcal{L}_0}(q_0) \gamma k \text{dim } \mathcal{O}_{kq_0} + O(\gamma^0 k^{n+1}). \tag{3.5} \]

Moreover, combining (3.3), (3.4) and (3.5), we have

Lemma 3.3.
\[ \text{Tr}(H^0(\mathcal{L}_0^k)) = \text{Tr}(H^0(\mathcal{L}_{\gamma,0}^k)) - \gamma \mu_{\mathcal{L}_0}(q_0) \frac{k^{n+1}}{n!} - \gamma \mu_{\mathcal{L}_0}(q_0) \frac{k^n}{2(n-2)!} + O(\gamma^0 k^{n+1}). \]

Now we are able to compute the Donaldson-Futaki invariant of \((\mathcal{X}, \mathcal{L})\). For this we need expansions

\[ h^0(\mathcal{L}_0^k) = a_0 \gamma^n k^n + a_1 \gamma^{n-1} k^{n-1} + O(\gamma^{n-2} k^{n-2}), \]

\[ \text{Tr}(H^0(\mathcal{L}_0^k)) = b_0 \gamma^{n+1} k^{n+1} + b_1 \gamma^n k^n + O(\gamma^{n-1} k^{n-1}), \]

\[ h^0(\mathcal{L}_{\gamma,0}^k) = a_0(\gamma) \gamma^n k^n + a_1(\gamma) \gamma^{n-1} k^{n-1} + O(\gamma^{n-2} k^{n-2}), \]

\[ \text{Tr}(H^0(\mathcal{L}_{\gamma,0}^k)) = b_0(\gamma) \gamma^{n+1} k^{n+1} + b_1(\gamma) \gamma^n k^n + O(\gamma^{n-1} k^{n-1}). \]

Once this notation is fixed, we can collect the results of the previous lemmas in the following

Corollary 3.1.
\[ a'_0(\gamma) = a_0 \gamma^n - \frac{1}{n!}, \]
\[ a'_1(\gamma) = a_1 \gamma^{n-1} - \frac{1}{2(n-2)!}, \]
\[ b_0(\gamma) = b_0 \gamma^{n+1} - \mu_{\mathcal{L}_0}(q_0) \frac{\gamma}{n!} + O(1), \]
\[ b'_1(\gamma) = b_1 \gamma^n - \mu_{\mathcal{L}_0}(q_0) \frac{\gamma}{2(n-2)!} + O(1). \]
Theorem 3.1 (Blow-up formula). Let \( F(X) \) and \( F(\hat{X}) \) denote the Donaldson-Futaki weights of \((X, L)\) and \((\hat{X}, \hat{L})\) respectively. Then, for \( \gamma \to \infty \), we have

\[
F(\hat{X}) = F(X) - \frac{1}{a_0^2} \left( \frac{b_0}{a_0} - \mu L_0(q_0) \right) \frac{\gamma^{1-n}}{2(n-2)!} + O(\gamma^{-n}).
\] (3.6)

Proof. The formula is an easy consequence of some elementary algebraic computations. We have

\[
b_0' a_1' = b_0 a_1 - b_0 \frac{\gamma^{n+1}}{2(n-2)!} + O(\gamma^n),
\]

\[
b_1' a_0' = b_1 a_0 - a_0 \mu L_0(q_0) \frac{\gamma^{n+1}}{2(n-2)!} + O(\gamma^n),
\]

\[
(a_0')^2 = a_0^2 \gamma^{2n} + O(\gamma^n).
\]

Thus,

\[
F(\hat{X}) = \frac{b_0' a_1' - b_1' a_0'}{a_0^2} = \frac{b_0 a_1 - b_1 a_0}{a_0^2} - \frac{1}{a_0} \left( \frac{b_0}{a_0} - \mu L_0(q_0) \right) \frac{\gamma^{1-n}}{2(n-2)!} + O(\gamma^{-n})
\]

Noting that \( F(X) = \frac{b_0 a_1 - b_1 a_0}{a_0^2} \) we can conclude the proof. \(\square\)

Remark 3.3. Note that we used our smoothness hypothesis only to give an explicit interpretation to the space of sections of \( O_{kq_0} \). All other arguments are exactly the same as the proof given in [26].

3.3 Orbifold blow-up and test configurations

In this section we want to extend the notions and the results of the previous two parts of this chapter, i.e. we want to show how to blow up a test configuration for a polarised orbifold and relate the Donaldson-Futaki invariant of this new test configuration to the weight of the original one.

Firstly, we have to make clear our definition of orbifold blow-up. To this purpose, let \((X, L)\) be a polarised orbifold and \( p \) a point in \( X \). By definition, there is an orbichart \((U, \Gamma, V)\) around \( p \). In particular, we can suppose that \( 0 \in U \subset \mathbb{C}^n \) is fixed by the action of \( \Gamma \) and maps onto \( p \), i.e. \( U/\Gamma \ni 0 \to p \in V \). One can then consider the blow-up \( \text{Bl}_0 U \) of the open set \( U \) at \( 0 \in U \). Since \( 0 \) is a fixed point for the action of \( \Gamma \) on \( U \), it lifts to an action on \( \text{Bl}_0 U \), i.e. \( \Gamma \act \text{Bl}_0 U \). Furthermore, we can cover \( \text{Bl}_0 U \) with affine open \( \Gamma \)-sets \((U_i)_{i \in I}\). Now, we define the orbifold blow-up of \( X \) at the point \( p \) and we denote it in the usual way \( \text{Bl}_p X \) as the orbifold obtained by gluing the orbicharts given by the open sets \((U_i)_{i \in I}\) and all other charts of the orbifold atlas \( \mathcal{U} \) of \( X \).
We use the same strategy to construct the blow-up of an orbifold at a general sub-orbischeme $Z$.

**Definition 3.1.** Let $(X, L)$ be a polarised orbifold. By sub-orbischeme $Z \subset X$ we mean an invariant sub-scheme $Z_U$ for each orbifold chart $U \rightarrow U/\Gamma \subset X$ such that, for each injection $U' \hookrightarrow U$, the sub-scheme $Z_{U'}$ is the scheme-theoretic intersection $Z_U \cap U'$.

Now consider a polarised orbifold $(X, L)$ and a sub-orbischeme $Z \subset X$. Locally, for each orbifold chart $(U, \Gamma, \phi)$, we can construct the blow-up $\text{Bl}_{Z_U} U$ of $U$ at the invariant sub-scheme $Z_U$ and, because $Z_U$ is invariant under the $\Gamma$-action, $\text{Bl}_{Z_U} U$ inherits the structure of $\Gamma$-space. Moreover, just as in the same way of the previous case, we can cover each $\text{Bl}_{Z_U}$ of affine open sets and glue them together to obtain a new orbifold, which we call the blow-up of $X$ at the sub-orbischeme $Z$, $\text{Bl}_Z X$. The exceptional divisors glue together to give an orbifold exceptional divisor $E \subset \mathcal{X}_0$.

Note that $\text{Bl}_Z X$ is well defined as an orbifold because, locally, we blow up invariant subscheme with the property of Definition 3.1.

Secondly, we proceed to define a certain class of orbifold test configurations, that we call *special*, for which we are able to extend the blow-up formula.

**Definition 3.2.** Let $(\mathcal{X}, \mathcal{L})$ be an orbifold test configuration for a polarised orbifold $(X, L)$. We say that $(\mathcal{X}, \mathcal{L})$ is special if both the total space $\mathcal{X}$ and the central fibre $\mathcal{X}_0$ are endowed with a (smooth) orbifold structure.

Let us consider a special test configuration $(\mathcal{X}, \mathcal{L})$ for a polarised orbifold $(X, L)$ and a point $p \in \mathcal{X}_1 \cong X$ in the fibre over $1$ and put $Z = (\mathbb{C}^* p)^-$, by which we mean the "orbi-schematic" closure of the orbit of the point $p$, i.e., for each orbifold chart $(U, \Gamma, \phi)$, $Z_U$ is defined as the schematic closure of $\phi^{-1}((\mathbb{C}^* p) \cap U/\Gamma)$. It is obvious that $Z_U$ is $\Gamma$-invariant and satisfies the property of Definition 3.1, thus $Z$ is indeed an orbi-subscheme of $\mathcal{X}$.

Then, just as in the manifold case, we can consider the orbifold $\mathcal{X}' = \text{Bl}_Z \mathcal{X}$, that is polarised by the orbi-line bundle $\mathcal{L}' = \pi^* \mathcal{L}^t \otimes \mathcal{O}(-E)$, where $\pi: \text{Bl}_Z \mathcal{X} \rightarrow \mathcal{X}$ is the blow-up map, with exceptional divisor $E$.

**Remark 3.4.** We refer to Proposition 3.1 for the proof that $(\mathcal{X}', \mathcal{L}')$ defines an orbifold test configuration for the polarised orbifold $(X, L)$, as, locally, we are constructing the same test configuration of the case considered in the previous section. The only part of the proof that deserves some further explanations is the relative ampleness of the orbi-line bundle $\mathcal{L}'$. More precisely, we have to show that the restriction $\mathcal{L}'_t = \mathcal{L}'|_{\mathcal{X}'_t}$ of $\mathcal{L}'$ to each fibre of the flat family $\mathfrak{a}: \mathcal{X} \rightarrow \mathbb{C}$ is ample. By Definition 1.15 we need to prove that $\mathcal{L}'_t = \pi_t^* \mathcal{L}_t \otimes \mathcal{O}(-E_t)$, where $E_t$ is the exceptional divisor of the blow-up $\pi_t: \text{Bl}_{E_t} \mathcal{X}_t \rightarrow \mathcal{X}_t$, is relatively ample and globally positive. The second condition is easily verified since, by the properties of the orbifold blow-up, $\text{ord}(\mathcal{L}_t) = \text{ord}(\mathcal{X}_t)$ and $\mathcal{L}'_t^{\text{ord}(\mathcal{X}_t)}$ is an honest line bundle, which is positive in the classical sense. Moreover,
\( \hat{L} \) is relatively ample because the stabiliser group \( \Gamma \) at a point \( x \) acts only on the first factor of the orbi-line \( (\hat{L})_x \), and this action is faithful by relative ampleness of \( L \).

Since the orbifold blow-up is obtained by a local construction, exactly as in the manifold case, we can understand the geometry of the central fibre of the test configuration \((\hat{X}, \hat{L})\): \( \hat{X}_0 = \text{Bl}_{q_0} X_0 \). This gives a proof of the following claim: if \((X, L)\) is a special test configuration for a polarised orbifold \((X, L)\), then the pair \((\hat{X}, \hat{L})\) constructed above is a special test configuration for \((\text{Bl}_q X, \pi^*L \otimes O(-E))\).

### 3.4 Blow-up formula for special test configurations

In this section we want to extend formula (3.6) to the case of special orbifold test configurations. The strategy of the proof is exactly the same as for manifolds, with some slight changes. The main point is to understand which arguments still hold true in the orbifold setting and which ones have to be modified.

Since orbi-line bundles are collections of equivariant line bundles on the orbifold charts that glue together, our general principle is that every statement for ordinary line bundles, which is local and that can be made equivariantly, can be extended to orbi-line bundles: exactness of a short exact sequence of invertible sheaves is an example of such statements.

**Remark 3.5.** Throughout this section the word orbifold will always mean orbifold with cyclic quotient singularities in codimension one. This assumption will be crucial in the proof of Proposition 3.2.

Recall that, in order to compute the Donaldson-Futaki invariant of an orbifold test configuration \((\hat{X}, \hat{L})\), we have to compute the following asymptotic expansions:

- \( h^0_{\hat{X}_0}(\hat{L}^k) = a_0k^n + a_1k^{n-1} + \delta(k^{n-1}) \),
- \( \text{Tr}(H^0(\hat{X}_0, \hat{L}_0^k)) = b_0k^{n+1} + b_1k^n + \delta(k^n) \),

and refer to Section 1.6 of the first chapter for the definition of \( \delta(k^n) \).

**Remark 3.6.** Following the general principle mentioned above, consider the sheaf isomorphism \( \pi_* O(\pi^{-1}(q_0))^{-k} = \mathcal{I}^k_{q_0} \) for \( k \gg 0 \) of Section 3.2. We claim that this is an isomorphism even in the category of orbi-sheaves. Indeed, on one hand we know how to define \( O(-D) \) for a divisor \( D \) in an orbifold; on the other hand, to define \( \mathcal{I}^k_{q_0} \) we proceed as follows: let \((U, \Gamma, \phi)\) be an orbifold chart around \( q_0 \), with \( U/\Gamma \ni 0 \mapsto q_0 \). Then, on \( U \), \( \mathcal{I}^k_{q_0}|_U \) is the ideal sheaf of holomorphic functions that vanish at 0. We glue it, along all other orbicharts of the atlas, with the (local) structure sheaf, obtaining the orbisheaf \( \mathcal{I}^k_{q_0} \). Since two sheaves being isomorphic is a local statement, it follows that, with our definition of \( \mathcal{I}^k_{q_0} \), \( \pi_* O(\pi^{-1}(q_0))^{-k} \cong \mathcal{I}^k_{q_0} \) for \( k \gg 0 \), as orbi-sheaves. Moreover, as in the manifold case, we have \( H^0(\text{Bl}_{q_0} X_0, \mathcal{I}_0) = H^0(\mathcal{X}_0, \mathcal{X}^k_{q_0}) \), for \( k \gg 0 \).
Keeping in mind the previous remark, we now proceed exactly as in Section 3.1. Indeed, consider the short exact sequence of invertible orbi-sheaves, given by

$$0 \to \mathcal{L}_0^{\gamma k} \mathcal{O}_{q_0}^k \to \mathcal{L}_0^{\gamma k} \to \mathcal{L}_0^{\gamma k}(q_0) \mathcal{O}_{kq_0}^0 \to 0,$$  

(3.7)

where, by $\mathcal{L}_0^{\gamma k}(q_0)$ we mean the fibre over 0 of the line bundle that defines $\mathcal{L}_0^{\gamma k}$, locally in an orbifold chart around $q_0$. Moreover, $\mathcal{O}_{kq_0}^0$ is the skyscraper orbi-sheaf of $\Gamma_{q_0}$-invariant jets of order up to $k - 1$ around 0, where $\Gamma_{q_0}$ is the stabiliser group at $q_0$. Just as in the manifold case, we can give a very explicit description of the space of sections of $\mathcal{O}_{kq_0}^0$: indeed, let $m = |\Gamma_{q_0}|$, so we have $\Gamma_{q_0} = \mathbb{Z}/m\mathbb{Z}$; then, by Example 1.13 of the first chapter, we can suppose that the orbifold chart around $q_0$ is of the form $(z_1, \ldots, z_n) \mapsto (z_1^n, \ldots, z_n)$, so $\mathbb{Z}/m\mathbb{Z}$ acts on $U = \mathbb{C}^n$ by multiplication for a primitive $m$-th root of unity. Thus, the space of sections of $\mathcal{O}_{kq_0}^0$ is made by all polynomials of degree up to $k - 1$, which are invariant under the action of $\mathbb{Z}_m$.

The only non-trivial step in the proof of exactness of sequence (3.7) is showing the surjectivity of $\mathcal{L}_0^{\gamma k} \to \mathcal{L}_0^{\gamma k}(q_0) \mathcal{O}_{kq_0}^0$: choose a section $s$ of $\mathcal{L}_0^{\gamma k}$ such that $s(q_0) \neq 0$; this means that there is an invariant section $s\mathcal{U}$ of the line bundle $(\mathcal{L}_0^{\gamma k})_U \to U$ such that $s\mathcal{U}(0) \neq 0$. Thus, every section $\tilde{s} \in H^0(\mathcal{L}_0^{\gamma k})$ can be written, locally in the orbichart $U$, as $\tilde{s}_U = f s\mathcal{U}_U$, where $f$ is a holomorphic function around 0 $\in U$. Furthermore, the fact that both $\tilde{s}_U$ and $s\mathcal{U}$ are invariant implies that $f$ is invariant as well. This proves exactness of (3.7).

The next result computes the asymptotic expansion of the dimension of the space of sections of $\mathcal{O}_{kq_0}^0$, as $k \to \infty$.

**Proposition 3.2.** Let $\mathcal{X}_m^k$ be the space of sections of $\mathcal{O}_{kq_0}^0$, then, for $k \to \infty$,

$$\dim \mathcal{X}_m^k = \frac{1}{n! m^n} k^n + \frac{m + n - 2}{2 m(n - 1)!} k^{n-1} + O(k^{n-2}).$$  

(3.8)

**Proof.** Since $\mathcal{X}_m^k = \{ P \in \mathbb{C}[z_1, \ldots, z_n]_{k-1} | P(\lambda z_1, \ldots, z_n) = P(z_1, \ldots, z_n) \}$, where $\lambda$ is a primitive $m$-th root of unity, a basis of such a space is given by monomials in $z_1, \ldots, z_n$ of the form: $z_1^{j_1} z_2^{j_2} \cdots z_n^{j_n}$, such that $jm + j_2 + \cdots + j_n \leq k - 1$. Moreover, the range of the index $j$ is $\{0, \ldots, \lfloor \frac{k-1}{m} \rfloor \}$ and, once $j$ is fixed, we must have $j_2 + \cdots + j_n \leq k - 1 - mj$, $j_i \geq 0$ for $i = 2, \ldots, n$ and the numbers of such $j$ is exactly the dimension of the space of polynomials in $n - 1$ variables of degree up to $k - 1 - mj$. Thus,

$$\dim \mathcal{X}_m^k = \sum_{j=0}^{\lfloor \frac{k-1}{m} \rfloor} \prod_{i=1}^{n-1} \frac{k - 1 - mj + i}{(n - 1)!} = \sum_{j=0}^{\lfloor \frac{k-1}{m} \rfloor} \frac{1}{(n - 1)!} \prod_{i=0}^{n-2} (k - mj + i).$$

Now we expand the product as

$$\prod_{i=0}^{n-2} (k - mj + i) = (k - mj)^{n-1} + (k - mj)^{n-2} \left( \frac{(n-1)(n-2)}{2} \right) + O((k - mj)^{n-3}),$$
which implies that
\[
\dim X^k_m = \frac{1}{(n-1)!} \sum_{j=0}^{k-1} (k-mj)^{n-1} + \frac{(n-2)(n-1)}{2(n-1)!} \sum_{j=0}^{k-1} (k-mj)^{n-2} + O(k^{n-2}),
\]
so we are left to asymptotically expand the following two terms:

\[
\vartheta_1(k) = \sum_{j=0}^{k-1} (k-mj)^{n-1},
\]
\[
\vartheta_2(k) = \sum_{j=0}^{k-1} (k-mj)^{n-2}.
\]

Binomially expanding the term \((k-mj)^t\), \(t = n-1, n-2\), we obtain

\[
\vartheta_1(k) = k^{n-1} + \sum_{j=1}^{k-1} \sum_{u=0}^{n-1} (-1)^u m^u j^u \left( \frac{n-1}{u} \right) k^{n-1-u} = k^{n-1} + \sum_{u=0}^{n-1} (-1)^u m^u k^{n-1-u} \left( \frac{n-1}{u} \right) \sum_{j=1}^{k-1} j^u.
\]

Next, use the Euler-Mclaurin formula to expand \(\sum_{j=1}^{k} j^u\) as \(\sum_{j=1}^{k} j^u = \frac{k^{u+1}}{u+1} + \frac{k^u}{u} + O(t^{-1})\), which in our case implies that

\[
\vartheta_1(k) = k^{n-1} + k^{n-1} \left( \left\lfloor \frac{k-1}{m} \right\rfloor + 1 \right) + \sum_{u=0}^{n-1} (-1)^u \left( \frac{n-1}{u} \right) m^u k^{n-1-u} \left( \frac{\left\lfloor \frac{k-1}{m} \right\rfloor + 1}{u+1} + \frac{\left\lfloor \frac{k-1}{m} \right\rfloor u}{2} + O(k^{n-1}) \right)
\]

and, writing \(\left\lfloor \frac{k-1}{m} \right\rfloor = \frac{k}{m} + \epsilon(k)\), where \(\epsilon(k)\) is a bounded periodic function in \(k\), we have

\[
\vartheta_1(k) = k^{n-1} \left( \frac{k}{m} + \epsilon(k) + 1 \right) + \frac{k^u}{m} \sum_{u=1}^{n-1} \frac{(-1)^u \left( \frac{n-1}{u} \right)}{u+1} + k^{n-1} \left( \epsilon(k) + \frac{1}{2} \right) \sum_{u=1}^{n-1} (-1)^u \left( \frac{n-1}{u} \right) + O(k^{n-2}).
\]

Finally, by recalling the properties of Newton binomial formula, we compute \(\sum_{u=1}^{n-1} \frac{(-1)^u \left( \frac{n-1}{u} \right)}{u+1} = \frac{1}{n-1} - 1\) and \(\sum_{u=1}^{n-1} (-1)^u \left( \frac{n-1}{u} \right) = -1\) and these identities enable us to conclude that

\[
\vartheta_1(k) = \frac{k^n}{mn} + \frac{k^{n-1}}{2} + O(k^{n-2}).
\]
With very similar computations, one can show that
\[ \vartheta_2(k) = \frac{k^{n-1}}{m(n-1)} + O(k^{n-2}); \]
thus, we have
\[ \dim \mathcal{X}_m^k = \frac{1}{(n-1)!} \vartheta_1(k) + \frac{(n-1)(n-2)}{2(n-1)!} \vartheta_2(k) + O(k^{n-2}) = \]
\[ \frac{1}{n!m} k^n + \frac{m+n-2}{2m(n-1)!} k^{n-1} + O(k^{n-2}), \]
which is exactly (3.8).

Remark 3.7. Note that (3.8) perfectly fits as an extension of expansion (3.4): indeed, \( \mathcal{X}_m^k = \mathbb{C}[z_1, \ldots, z_n]_{k-1} \) and (3.8) coincides with (3.4).

In order to obtain the orbifold blow-up formula, following the strategy of the proof in Section 3.2, we should now use a vanishing result for the higher cohomology of sequence (3.7): this is a subtle problem, since it is not even obvious how to define the higher cohomology groups of an orbi-sheaf over a smooth orbifold. A possibility is to consider orbifolds as a special class of stacks and to define their cohomology with the existing theory of stack cohomology, for an exhaustive exposition of which we refer to \cite{19} and \cite{24}.

Once we have the notion of \( H_i(X, L) \) for every \( i > 0 \), where \( X \) is an orbi-scheme and \( L \) a sheaf on \( X \), in analogy with the classical theory, we need a notion of ampleness for a locally free orbi-sheaf on a stack, such that one can extend the vanishing results known for positive (ample) vector bundles on compact complex manifolds.

Note that, by Remark 1.12 of the first chapter, when \( X \) is a smooth orbifold and \( L \) an orbi-line bundle on \( X \), the notion of ampleness is equivalent to the differential geometric notion of positivity. Thus, a possible way to prove the vanishing result needed is to extend the classical Bochner technique, see, for example, \cite{8}; actually, since we allow the central fibres of orbifold test configurations to have an orbi-scheme structure, i.e., they can be singular orbifolds, we need a vanishing theorem which applies to the more general context on orbi-schemes. More precisely, we need \( H^i(L^k \otimes \mathcal{F}) = 0 \) for \( i > 0 \) for any coherent sheaf and for \( k \gg 0 \), given that we want to use this result when \( \mathcal{F} = \mathcal{I}_{q_0} \).

Conjecturally, this could be taken as definition on ampleness of an orbi-line bundle on an orbi-scheme - indeed this condition, in the context of manifold, is a characterisation of ampleness of line bundles. Furthermore, we make the following assumption

Assumption 3.1. The notion of ampleness for line bundles \( L \) on orbi-schemes used in \cite{20} actually implies the vanishing \( H^i(L^k \otimes \mathcal{F}) = 0, i > 0 \), for all coherent \( \mathcal{F} \) and sufficiently large \( k \), for our choice of stack cohomology theory \( H^* \).

Now consider the long exact cohomology sequence canonically associated to (3.7). By the previous remark, we can assume that, for \( k \gg 0 \), independently of \( \gamma > \gamma_0 \), \( H^1(\mathcal{E}_0, \mathcal{L}_0^k \mathcal{J}_{q_0}^k) = \)
In order to compute the orbifold Donaldson-Futaki invariant we need the following expansions:

\[ 0 \to H^0_{\mathcal{X}_0}(\mathcal{L}^k_{\mathcal{O}_{\mathcal{X}_0}}) \to H^0(\mathcal{L}^k_{\mathcal{O}_{\mathcal{X}_0}}) \to O^{\Gamma_0}_{kq_0} \otimes \mathcal{L}^k_{\mathcal{O}_{\mathcal{X}_0}}(q_0) \to 0. \quad (3.9) \]

Note that, at least formally, sequences (3.9) and (3.1) are exactly the same. Thus, we can use the arguments and the techniques of Section 3.2 to extend formula (3.6) to the case of special test configurations. In particular, the two following identities hold true for \( k \gg 0 \):

\[ h^0(\mathcal{X}_0, \mathcal{L}^k_{\mathcal{O}_{\mathcal{X}_0}}) = h^0_{\mathcal{O}_{\mathcal{X}_0}} - \dim O^{\Gamma_0}_{kq_0} \otimes \mathcal{L}^k_{\mathcal{O}_{\mathcal{X}_0}}(q_0), \quad (3.10) \]

\[ \text{Tr}(H^0_{\mathcal{X}_0}(\mathcal{L}^k_{\mathcal{O}_{\mathcal{X}_0}})) = \text{Tr}(H^0(\mathcal{L}^k_{\mathcal{O}_{\mathcal{X}_0}})) - \text{Tr}(O^{\Gamma_0}_{kq_0} \otimes \mathcal{L}^k_{\mathcal{O}_{\mathcal{X}_0}}(q_0)). \quad (3.11) \]

By combining formula (3.8) with (3.10) we can compute the first of the two asymptotic expansions needed in the definition of the orbifold Donaldson-Futaki invariant.

**Lemma 3.4.**

\[ h^0_{\mathcal{O}_{\mathcal{X}_0}} = h^0(\mathcal{L}^k_{\mathcal{O}_{\mathcal{X}_0}}) - \alpha k^n - \beta k^{n-1} + O(k^{n-2}), \]

where \( \alpha = \frac{1}{m} \) and \( \beta = \frac{m+n-2}{2m(n-1)^2} \).

Now we use (3.11) to compute the trace of the \(|\mathbb{C}|^*\)-action on the space \( H^0(\mathcal{L}^k_{\mathcal{O}_{\mathcal{X}_0}}) \). The only difference with the manifold case is that \( \text{Tr}(\mathcal{L}(q_0)) \) no longer has a straightforward algebro-geometric interpretation and we will denote this weight just by \( \mu \).

As in (3.5), we can write

\[ \text{Tr}(O_{kq_0} \otimes \mathcal{L}^k_{\mathcal{O}_{\mathcal{X}_0}}(q_0)) = \gamma k \dim O_{kq_0} + O(\gamma^0 k^{n+1}), \quad (3.12) \]

obtaining the orbifold analogue of Lemma 3.3.

**Lemma 3.5.**

\[ \text{Tr}(H^0_{\mathcal{O}_{\mathcal{X}_0}}(\mathcal{L}^k_{\mathcal{O}_{\mathcal{X}_0}})) = \text{Tr}(H^0(\mathcal{L}^k_{\mathcal{O}_{\mathcal{X}_0}})) - \gamma \mu \alpha k^{n+1} - \gamma \mu \beta k^n + O(\gamma^0 k^{n+1}). \]

From now on, we can repeat exactly the steps of Section 3.2.

In order to compute the orbifold Donaldson-Futaki invariant we need the following expansions:

\[ h^0(\mathcal{L}^k_{\mathcal{O}_{\mathcal{X}_0}}) = a_0 \gamma^n k^n + a_1 \gamma^{n-1} k^{n-1} + \delta(\gamma^{n-2} k^{n-2}), \]

\[ \text{Tr}(H^0(\mathcal{L}^k_{\mathcal{O}_{\mathcal{X}_0}})) = b_0 \gamma^{n+1} k^{n+1} + b_1 \gamma^n k^n + \delta(\gamma^{n-1} k^{n-1}), \]

\[ h^0(\mathcal{L}^k_{\mathcal{O}_{\mathcal{X}_0}}) = a'_0(\gamma) \gamma^n k^n + a'_1(\gamma) \gamma^{n-1} k^{n-1} + \delta(\gamma^{n-2} k^{n-2}), \]

\[ \text{Tr}(H^0(\mathcal{L}^k_{\mathcal{O}_{\mathcal{X}_0}})) = b'_0(\gamma) \gamma^{n+1} k^{n+1} + b'_1(\gamma) \gamma^n k^n + \delta(\gamma^{n-1} k^{n-1}). \]

As a consequence of the previous computations we obtain the following
Corollary 3.2.

\[ a'_0(\gamma) = a_0 \gamma^n - \alpha, \]
\[ a'_1(\gamma) = a_1 \gamma^{n-1} - \beta, \]
\[ b'_0(\gamma) = b_0 \gamma^{n+1} - \mu \gamma \alpha + O(1), \]
\[ b'_1(\gamma) = b_1 \gamma^n - \mu \gamma \beta + O(1). \]

Finally, we can state the orbifold blow-up formula, which is the central topic of this chapter.

Theorem 3.2 (Orbifold Blow-up formula). Let \( F(X) \) and \( \hat{F}(\hat{X}) \) denote the orbifold Donlandson-Futaki weight of \((X, L)\) and \((\hat{X}, \hat{L})\) respectively. Then, for \( \gamma \to \infty \), we have

\[ F(\hat{X}) = F(X) - \frac{1}{a_0} \left( \beta \frac{b_0}{a_0} - \mu \alpha \right) \gamma^{1-n} + O(\gamma^{-n}), \quad (3.13) \]

where \( \alpha = \frac{1}{n!} \) and \( \beta = \frac{m+n-2}{2m(n-1)!} \).

Proof. Repeat verbatim the proof of Theorem 3.1.

Remark 3.8. Note that Theorem 3.2 applies not only to special test configurations. Indeed, the core of the proof of the orbifold blow-up formula relies on Proposition 3.2: in order to understand the structure of the orbi-sheaf \( O_{\Gamma q_0}^X \) and its space of sections we do not need to know about the global geometry of the central fibre \( X_0 \), but just the behaviour of \( q_0 \) as a point in \( X_0 \); in particular, \( q_0 \) has to be a smooth orbifold point. Since smoothness of a point on an orbifold is a local statement, we can allow \( X_0 \) to be singular away from the point \( q_0 \) and, in such a case, all the statements and proofs of this section still hold true.

Now, in order to obtain a complete extension of Theorem 2.8 following the proof given in \( [25] \), we should proceed as follows: we need a result ensuring that, exactly as in the manifold case, we can find a point \( q \) such that the quantity \( C(q) := \beta \frac{b_0}{a_0} - \mu \alpha \) is strictly positive. For manifolds, this is proved in the following way: after a base change of the form

\[ \rho : \mathbb{C} \to \mathbb{C}, \quad z \mapsto z' \]

for the test configuration, \( C \) is seen, modulo a positive constant, as the Chow weight of \( q_0 \) under the projective embedding given by powers of \( \rho^* L_0 \). Moreover, in \( [25] \) it is proved that one can choose \( q \) such that the aforementioned Chow weight of the limit point \( q_0 \) is positive.

In the orbifold setting there is an obstruction that prevents us from extending verbatim this proof: we do not have an analogue of the Chow weight for a point in a weighted projective space. Thus, in order to complete the proof of the previous statement, we should define the orbi-Chow weight of a point in a weighted projective space and prove the following two intermediate results:
1. modulo a base change \( \rho \), the quantity \( C(q) \) can be seen as the \textit{orbi-Chow weight} of the point \( q_0 \) under the weighted projective embedding given by the orbi-line bundle \( \rho^*L_0 \);

2. the point \( q \) can be chosen so that the limit point \( q_0 \) has strictly positive orbi-Chow weight.

Since proving these results is still a work in progress, we will present them as a conjecture.

**Conjecture 3.1.** The point \( q \) can be chosen such that the quantity \( C(q) \) defined above is strictly positive.

Moreover, the second main result needed in order to extend Theorem 2.8 is an orbifold analogue of the Arezzo-Pacard Theorem. In other words, we need to know that, just as for manifolds, the problem of lifting a conical cscK metric from a compact cyclic orbifold \( X \) to the blow-up \( Bl_pX \) at point \( p \) with non-trivial stabiliser is unobstructed. Note that, in the general case, it is not obvious which should be the formal statement of this result. We can reach this goal in the case of orbifolds with cyclic quotient singularities in codimension one. In this setting, we formulate the following conjecture.

**Conjecture 3.2 (Arezzo-Pacard Theorem for cyclic orbifolds in codimension one).** Let \( X \) be a compact complex manifold and \( D \subset X \) a smooth divisor. Put a \( \mathbb{Z}_m \) stabiliser on \( D \), consider the orbifold \( X = (X, D) \) together with an ample orbi-line bundle \( L \) on \( X \). Suppose that \( X \) admits a cscK metric, conical along \( D \) with cone angle \( \frac{2\pi}{m} \), and that \( \text{Aut}(X) \) is discrete. Fix a point \( p \in D \). Then the following statements hold true:

- \( \text{Bl}_pX \) admits a cyclic orbifold structure of order \( m \) along the divisor \( \tilde{D} = \pi^{-1}(D) \setminus E \), i.e. the strict transform of \( D \), where \( \pi : \text{Bl}_pX \to X \) is the blow-up projection with exceptional divisor \( E \);
- there exists \( \varepsilon > 0 \) such that the cohomology class \( c_1(L) - \varepsilon E \) contains a cscK metric, conical along \( \tilde{D} \) of angle \( \frac{2\pi}{m} \).

Furthermore, if \( p \) is a smooth point, i.e. \( p \notin D \), there exists \( \delta > 0 \) such that the cohomology class \( c_1(L) - \delta E \) contains a cscK metric, conical along \( D \) of angle \( \frac{2\pi}{m} \).

### 3.5 Slope-stability for orbifolds

In this section we introduce another notion of stability for polarised orbifolds, which is slope-stability \cite{20}. Moreover, we will show how this notion is related with K-stability and investigate some possible applications of the orbifold blow-up formula. Although slope-stability is only a very special case of K-stability, it is especially important in the orbifold case because of its applications to Sasaki-Einstein metrics, \cite{31}.

The main tool needed to define the notion of slope-stability is a construction called \textit{deformation to the normal cone}. Thus, fix an \( n \)-dimensional polarised orbifold \((X, L)\) and...
a sub-orbischeme \( Z \subset X \) - recall Definition 3.1. In the previous section we extended the blow-up construction to the orbifold category, so let \( \pi : \text{Bl}_Z X \to X \) be the blow-up projection, with exceptional divisor \( E \). We know that for large \( N, \pi^* L^N(-E) \) is positive. Thus, we can define the Seshadri constant by

\[
\varepsilon_{\text{orb}}(Z) = \sup \{ x \in \mathbb{Q}_+ : (L(-xE))^M \text{ is ample for some } M \in \mathbb{N} \}.
\]

To get a test configuration from \( Z \) consider the sub-orbifold \( Z \times \{0\} \subset X \times C \). Blowing this up gives a degeneration \( \mathcal{X} \to X \times C \to C \) which is called degeneration to the normal cone of \( Z \) with exceptional divisor \( P \). As shown in [21] for schemes (and the same result goes through easily for orbifolds), \( \varepsilon_{\text{orb}}(Z \times \{0\}) = \varepsilon_{\text{orb}}(Z) \). Then, for generic \( c \in (0, \varepsilon_{\text{orb}}(Z)) \cap \mathbb{Q} \), integer powers of \( L_\epsilon := p^\epsilon L(-cP) \), where \( p : \mathcal{X} \to X \) is the projection, define a polarisation on \( \mathcal{X} \). The natural action of \( C^* \) on \( X \times C \), which is trivial on \( (X, L) \) and with weight one on \( C \), lifts naturally to a linearised action on \( (\mathcal{X}, \mathcal{L}) \) and thus for such \( c \) we have a test configuration \( (\mathcal{X}, \mathcal{L}) \) with general fibre \( (X, L) \). The central fibre is \( \mathcal{X}_0 = \text{Bl}_Z X \cup_E P \) consisting of the blow-up \( \text{Bl}_Z X \) along \( Z \) glued to \( P \) along \( E \) and the induced \( C^* \)-action is trivial on \( \text{Bl}_Z X \) and acts by scaling \( P \) along \( E \) and the normal to \( E \).

As usual we write

\[
h^0 (L^k) = a_0 k^n + a_1 k^{n-1} + \delta(k^{n-1}),
\]

and then define the slope of \( (X, L) \) to be

\[
\mu(X,L) = \frac{a_1}{a_0} = -\frac{n \int_X c_1(K_{\text{orb}})c_1(L)^{n-1}}{2\int_X c_1(L)^n},
\]

where the second equality comes from formula (1.2). Now we want to define the slope of \( Z \subset X \) and in order to do this we work on the orbifold blow-up \( \pi : \text{Bl}_Z X \to X \) along \( Z \) with exceptional divisor \( E \). Then the orbifold Riemann-Roch Theorem applied to \( \pi^* L^k(-\frac{j}{k}E) \) for fixed \( j \) (and \( k = jK \) for some integers \( K \)) takes the form

\[
h^0 (\pi^* L^k(-\frac{j}{k}E)) = p(k,j) + \varepsilon_p(k,j),
\]

where \( p \) is a polynomial of two variables of total degree \( n \) and \( \varepsilon_p \) is a sum of terms of the form \( r_p \delta' \) for some polynomial \( r_p \) of two variables of total degree \( n-1 \) and a function \( \delta' = \delta'(k,j) \) periodic in each variable and with average \( \sum_{k,j=1}^M \delta'(k,j) = 0 \). Define then polynomials \( a_i(x) \) by

\[
p(k,xk) = a_0(x)k^n + a_1(x)k^{n-1} + O(k^{n-2}) \text{ for } kx \in \mathbb{N}
\]

and extend by the same formula for \( x \in \mathbb{R} \). Then the slope of \( Z \) (with respect to \( c \)) is defined as the quantity

\[
\mu_c(\mathcal{X}_Z) := \frac{\int_0^c a_1(x) + \frac{\delta'(x)}{2}}{\int_0^c a_0(x)} dx. \quad (3.14)
\]
Remark 3.9. Comparing the previous definition with the one given in [21], we see that the only difference with the manifold case is that we ignored the periodic terms in the relevant Hilbert functions. This amounts to replacing $K_X$ by $K_{\text{orb}}$.

Definition 3.3. We say that a polarised orbifold $(X, L)$ is slope-semistable with respect to a sub-orbischeme $Z$ if

$$\mu(I_Z) \leq \mu(X) \text{ for all } 0 < c < \varepsilon_{\text{orb}}(Z).$$

We say that $X$ is slope-semistable if it is slope-semistable with respect to all sub-orbischemes $Z \subset X$.

Remark 3.10. Note that if in formula (3.15) we require the inequality to be strict, then we have the notion of slope-stable orbifold.

The connection between slope-stability and K-stability (both in the manifold and orbifold settings) is given by a result by Ross and Thomas: the main point is that the sign of the Futaki invariant of the test configuration given by the deformation to the normal cone of $Z$ is the same as the sign of $\mu(X) - \mu_c(I_Z)$. More precisely, we have the following result, for a proof of which we refer to [20].

Theorem 3.3. Let $(X, L)$ be a polarised orbifold, $Z \subset X$ a sub-orbischeme of $X$ and $(\mathcal{X}_Z, \mathcal{L}_c)$ the test configuration given by the deformation to the normal cone of $Z$. Then, with the above definitions and notations, the following equality holds true:

$$F(\mathcal{X}_Z, \mathcal{L}_c) = (\mu(X) - \mu_c(\mathcal{I}_Z)) \int_0^c \frac{a_0(x)dx}{a_0}$$

(3.16)

We know that $\int_0^c \frac{a_0(x)dx}{a_0}$ is non-negative, so if in addition we suppose that $(X, L)$ is K-semistable, we have that, for every sub-orbischeme $Z$, $F(\mathcal{X}_Z, \mathcal{L}_c)$ is non-negative, implying that $\mu_c(\mathcal{I}_Z) \leq \mu(X)$. This means that slope-stability comes as an obstruction to K-stability; indeed, we have the following

Corollary 3.3. If $(X, L)$ is K-semistable then it is slope-semistable.

We now want to show how, under certain hypotheses and assumptions, the orbifold blow-up formula can be used to strengthen the previous result. Our strategy is to use an argument which is completely analogous to the one presented in the previous section after Remark 3.8.

Firstly, we note that formula (3.13) applies when $\mathcal{X}$ is the deformation to the normal cone. Indeed, the total space of such test configuration is an orbifold; moreover, even if the central fibre $\mathcal{X}_0$ of $\mathcal{X}$ is not a smooth orbifold, the limit point $q_0$ is smooth in $\mathcal{X}_0$ for every choice of $q \in X = \mathcal{X}_1$. Thus, the deformation to the normal cone is not a special test configuration because of the singularities of the central fibre, but, by Remark 3.8, the orbifold blow-up formula still holds true. This enables us to investigate the possibility of finding a proof of Conjecture 3.1 in the special case of the deformation to the normal cone. To reach this goal we might find inspiration by considering the problem.
in the manifold setting: thus, let \((X, L)\) be a polarised manifold and \(Z \subseteq X\) a closed subscheme of \(X\); consider the test configuration \((\mathcal{X}, \mathcal{L})\) generated by \(Z\), where, in particular, \(\mathcal{X} = \text{Bl}_{Z \times \{0\}} X \times \mathbb{C}\). Then, it can be proved (for more details see [33]) that, if we choose \(q \in Z\), the limit \(q_0 := \lim_{t \to 0} t.q\) of \(q\) under the \(\mathbb{C}^*\)-action has negative Chow weight. Thus, in the manifold case we know exactly which are the points that we need to choose to construct destabilising test configurations. We conjecture that this result still holds true if we consider a polarised orbifold, but the proof of such an extension is still a work in progress.

The last tool needed is the orbifold analogue of Arezzo-Pacard Theorem, which we have stated as a conjecture in a previous section, since its proof seems far from being a trivial extension of the classical Arezzo-Pacard Theorem.

These arguments enable us to give a partial proof of the following result.

**Theorem 3.4.** Let us suppose that Conjectures 3.1 and 3.2 are true. Then, if a polarised orbifold with cyclic quotient singularities in codimension one and discrete automorphism group is cscK, it is slope-stable.

**Proof.** By contradiction, let us suppose that \(X\) is not slope-stable. This means that there is a sub-orbischeme \(Z\) such that the associated test configuration \((\mathcal{X}_Z, \mathcal{L}_c)\) satisfies

\[
\mu(X) = \mu_c(Z),
\]

for some \(c \in (0, \varepsilon_{\text{orb}})\). Moreover, formula (3.16) implies the vanishing of the Donaldson-Futaki invariant of \(\mathcal{X}_Z, F(\mathcal{X}_Z, \mathcal{L}_c) = 0\). Now by Conjecture 3.1, we can choose a point \(q \in \mathcal{X}_0\) such that the number \(C(q)\) in formula (3.13), applied to \(\mathcal{X}_Z\), is strictly positive. But then, for \(\gamma \gg 0\), \(F(\mathcal{X}_Z) < 0\), which implies that \(\text{Bl}_q X\) with its canonical polarisation is K-unstable. This is a contradiction since, by Conjecture 3.2, \(\text{Bl}_q X\) admits a conical cscK metric and thus is K-semistable by Corollary 2.3. \(\square\)
Acknowledgements

Questa tesi non avrebbe mai visto la luce senza l’aiuto ed il supporto del mio relatore, il Professor Jacopo Stoppa. A lui vanno i miei più sentiti ringraziamenti, per la pazienza con cui ha seguito questo lavoro, per la disponibilità dimostratami e per i tanti insegnamenti, matematici e non, che lungo tutto questo periodo mi ha trasmesso.

My heartfelt acknowledgements also go to Professor Julius Ross, for his invaluable support during my stay at the University of Cambridge, for helping me with the proof Theorem 3.2 and for providing me with an exceptional academic environment.

Voglio ringraziare tutti coloro che hanno reso unica la mia esperienza di vita universitaria a Pavia negli ultimi cinque anni.

Grazie alle mie coinquiline: a Simona, per le indimenticabili serate insieme, per le "pause coffee" e per l’amicizia che, soprattutto nell’ultimo anno, mi ha saputo dimostrare; a Francesca, per avermi dimostrato, seppur a suo modo, di volermi bene; a Valentina, per le colazioni sempre pronte; a Marta, che pur non essendo stata mia coinquilina a tutti gli effetti, è ormai di casa e mi ha saputo spesso far sorridere; a Flavia, per la breve ma intensa e soprattutto piacevolissima convivenza.

Grazie a Maddalena, senza il cui aiuto e supporto questi ultimi due anni sarebbero stati di certo molto più difficili. Per essere stata la spalla su cui piangere ma anche una compagna di risate, per essere un’amica su cui posso sempre contare, per le passeggiate sul Ticino, per aver sempre creduto in me, per non essersi mai arresa nella sua strenua battaglia contro il mio pessimismo cosmico, per avermi soportato nei miei momenti più bui ed aver condiviso con me i momenti di felicità.

Grazie a Vittoria, per avermi insegnato che l’amicizia supera anche i litigi più duri, per avermi sempre capito, per i preziosi consigli, per gli infiniti aperitivi trascorsi a parlare di tutto e di più, per avermi sempre aiutato a guardare le cose in modo positivo, per essere stata sempre schietta e pacata.

Grazie a Chiara, per la simpatia, l’affetto, per le impareggiabili imitazioni e scenette, per essere Chiara da Telecom.

Grazie a Guendalina, per le serate insieme, per la sua esplosiva e contagiosa voglia di vivere.

Grazie anche a tutte le altre persone che ho avuto il piacere di conoscere lungo questo percorso e che sono entrate, ognuna a modo loro, a far parte della mia vita: Fabio, Milo,
Francesco, Caterina, Giammarco.
Questi anni a Pavia sono stati indimenticabili e lasciare questa città vorrà dire per me lasciare qui anche un pezzo di cuore.

Grazie a Vanessa, per essere la dimostrazione vivente che l’Amicizia è uno dei valori più importanti. Per essere una delle mie poche certezze, perché so che lei ci sarà sempre, perché la distanza non ci ha mai diviso e mai ci dividerà, per essere stata di fondamentale aiuto pur essendo fisicamente lontano, per essere la mia migliore amica di sempre.

Grazie a Federico, perché dopo tutti questi anni siamo ancora qui, a volerci bene ed a sostenerci l’uno con l’altro.

Il rigraziamento più grande va alla mia famiglia.
Grazie alla Nonna Lucia, per essere la mia sostenitrice numero uno, per volere sempre il meglio per me.
Grazie a Paola e Manuel, a Gabriele, Francesca e Gaia, perché, anche se non ce lo diciamo, ci vogliamo bene e ci saremo sempre gli uni per gli altri.
Grazie ai miei genitori, Antonietta e Lucio, a cui questa tesi è dedicata, per essere la mia forza, per aver sempre cercato di capirmi, anche quando facevo di tutto per non essere compreso, per aver saputo accettarmi, per avermi sempre sostenuto, per aver sempre creduto in me e nelle mie scelte, per avermi dato tantissime opportunità, senza le quali non avrei mai potuto raggiungere questo traguardo, per trasmettermi tacitamente ma quotidiamente il loro amore per me. Sono onorato di avere dei genitori così.
Bibliography


