Index of fibrations and Brauer classes that never obstruct the Hasse principle

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Hasse Principle

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Hasse principle

A collection $\mathcal{C}$ of varieties is said to satisfy the Hasse principle if

$$X(\mathbb{A}) \neq \emptyset \implies X(k) \neq \emptyset$$

for all $X \in \mathcal{C}$.

Brauer–Manin obstruction

Lind-Reichardt (1941): $2z^2 = x^4 - 17y^4$ fails the Hasse principle.

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$\mathrm{Br}(X) := \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)_{\text{tors}}$ Brauer group of X. Brauer–Manin pairing:

$$\mathrm{ev}\colon \mathrm{Br}(X) \times X(\mathbb{A}) \to \mathbb{Q}/\mathbb{Z}.$$

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For each subset $H \subseteq \mathrm{Br}(X)$ one can define the obstruction set

$$X(\mathbb{A})^H := \{P \in X(\mathbb{A}) \mid \mathrm{ev}(\mathcal{A}, P) = 0 \ \forall \mathcal{A} \in H\}$$

such that

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If $X(\mathbb{A})^H = \emptyset$ but $X(\mathbb{A}) \neq \emptyset$ then we say there is a *Brauer–Manin obstruction* (to the Hasse principle) given by H.

Properties of the obstruction sets

If $H_1 \subseteq H_2$ then $X(\mathbb{A})^{H_1} \supseteq X(\mathbb{A})^{H_2}$.

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Define $\mathrm{Br}_0(X) := \mathrm{im}(\mathrm{Br}(k) \to \mathrm{Br}(X))$ (constant algebras).

$$X(\mathbb{A})^{\mathrm{Br}_0(X)} = X(\mathbb{A})$$

One often considers the quotient $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ for Brauer–Manin obstructions.

Investigating the obstruction

Challenges in computing $X(\mathbb{A})^{\mathrm{Br}}$:

- The quotient $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ can be large.
- Need to compute evaluation maps $\mathrm{ev}(\mathcal{A}, -)$ for any $\mathcal{A} \in \mathrm{Br}(X)$.

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Many counterexamples to the Hasse principle explained by a Brauer–Manin obstruction require only one element $\mathcal{A} \in \mathrm{Br}(X)$, i.e., $X(\mathbb{A})^{\mathcal{A}} = \emptyset$.

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Question

Is there a subset $H \subset \mathrm{Br}(X)$ that is irrelevant to the Brauer–Manin obstruction?

Investigating the obstruction

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Question

Given a class $\mathcal{C}$ of varieties over k, does there exist $n \in \mathbb{Z}$ such that $\mathrm{Br}(X)[n^\perp]$ never gives a Brauer–Manin obstruction? i.e., $X(\mathbb{A}) \neq \emptyset \implies X(\mathbb{A})^{\mathrm{Br}(X)[n^\perp]} \neq \emptyset$?

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Given a class $\mathcal{C}$ of varieties over k, does there exist $n \in \mathbb{Z}$ such that $\mathrm{Br}(X)[n^{\perp}]$ never gives a Brauer–Manin obstruction? i.e., $X(\mathbb{A}) \neq \emptyset \implies X(\mathbb{A})^{\mathrm{Br}(X)[n^{\perp}]} \neq \emptyset$?

A similar question ($X(\mathbb{A})^{\mathrm{Br}} = \emptyset \implies X(\mathbb{A})^{\mathrm{Br}(X)[n^{\infty}]} = \emptyset$?) was asked by Creutz and Viray (2017) where they focused on the case where n is the degree of the variety.

For cubic surfaces, we have the following answer

Theorem (Swinnerton-Dyer 1993)

Let X be a smooth cubic surface over a number field k. The possibilities for $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ are

$$\{1\}, \mathbb{Z}/3\mathbb{Z}, (\mathbb{Z}/3\mathbb{Z})^2, \mathbb{Z}/2\mathbb{Z}, (\mathbb{Z}/2\mathbb{Z})^2.$$

Moreover, $X(\mathbb{A}) \neq \emptyset \implies X(\mathbb{A})^{\mathrm{Br}(X)[3^\perp]} \neq \emptyset$.

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Other rational surfaces?

Rational surfaces

The question is invariant under birational morphisms for smooth projective surfaces: $X \xrightarrow{bir} Y$ then $X(\mathbb{A})^{\mathrm{Br}(X)[n^\perp]} \neq \emptyset \iff Y(\mathbb{A})^{\mathrm{Br}(Y)[n^\perp]} \neq \emptyset$.

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So consider minimal rational surfaces. We can classify minimal rational surfaces over a number field into the following:

1. Quadric surfaces
2. Conic bundle over a rational curve
3. Del Pezzo surfaces of degree $1 \leq d \leq 9$

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When X is either (1) or (2) above, it is well known that $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ is a 2-torsion group. So $\mathrm{Br}(X)[2^\perp]$ does not give a Brauer–Manin obstruction (trivially).
Note that here X has an ample divisor of even degree.

del Pezzo surfaces

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- $(d = 1)$ Always has a rational point

In the remaining case of del Pezzo surfaces of degree 2, the possible $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ are as follows (Corn 2007):

$$\{1\}, \mathbb{Z}/3\mathbb{Z}, (\mathbb{Z}/3\mathbb{Z})^2, (\mathbb{Z}/2\mathbb{Z})^s (1 \leq s \leq 6),$$
$$\mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^t (0 \leq t \leq 2), (\mathbb{Z}/4\mathbb{Z})^2$$

Main result

The *index* of a variety X/k is the gcd of all degrees of extensions K/k where X has a K-point.

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We say that a variety Y/k *satisfies property (ZC)* if for any field extension K/k and $Q \in Y(K)$, the natural map $Y(K) \to A_0(Y_K)$ given by $P \mapsto (P) - (Q)$ is surjective. E.g., smooth projective curves of genus 1 and k-rational varieties.

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Theorem (N 2017)

Let $\pi \colon X \to Z$ be a morphism of smooth projective geometrically integral varieties over a number field k. Suppose Z satisfies weak approximation and a Zariski dense set of the fibers of π satisfy (ZC). Suppose that the generic fiber over $k(Z)$ has index d. If $B \subset \mathrm{Br}(X)$ is a subset such that $X(\mathbb{A})^B \neq \emptyset$, then $X(\mathbb{A})^{B+\mathrm{Br}(X)[d^\perp]} \neq \emptyset$.

Applications

Any deg 2 del Pezzo X over number field k can be given by equation

$$w^2 = f(x, y, z)$$

in $\mathbb{P}[2, 1, 1, 1]$, where $f \in k[x, y, z]$ is homogeneous quartic.

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$$\begin{array}{ccc} \widetilde{X} & & \\ \downarrow \beta & \searrow \widetilde{\pi} & \\ X & \overset{\pi}{\dashrightarrow} & \mathbb{P}^1 \end{array}$$

$\pi([w : x : y : z]) = [y : z]$
β blow up along the π-indeterminacy.
$\widetilde{\pi}$ genus 1 fibration with generic fiber index $d \mid 2$.

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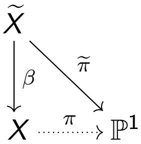

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Corollary

X deg 2 del Pezzo surface over number field k, then
$X(\mathbb{A}) \neq \emptyset \implies X(\mathbb{A})^{\mathrm{Br}(X)[2^\perp]} \neq \emptyset.$

Applications

A smooth diagonal quartics in $\mathbb{P}^3_k$ is defined by

$$ax^4 + by^4 + cz^4 + dw^4 = 0,$$

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Theorem (Ieronymou-Skorobogatov 2015)

Let X be a smooth diagonal quartic over $\mathbb{Q}$. Then $X(\mathbb{A}) \neq \emptyset \implies X(\mathbb{A})^{\mathrm{Br}(X)[2^\perp]} \neq \emptyset$.

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Corollary

Let X be a smooth diagonal quartic over a number field k, with $abcd \in k^{\times 2}$. If $B \subset \mathrm{Br}(X)$ is a subgroup such that $X(\mathbb{A})^B \neq \emptyset$, then $X(\mathbb{A})^{B+\mathrm{Br}(X)[2^\perp]} \neq \emptyset$.

Main result

k number field.

Theorem (Creutz-Viray 2017)

Let X be a k-torsor under an abelian variety, let $B \subset \mathrm{Br}(X)$ be any subgroup, and let d be the period of X. In particular, d could be taken to be the degree of a k-rational globally generated ample line bundle. If $X(\mathbb{A})^B \neq \emptyset$ then $X(\mathbb{A})^{B+\mathrm{Br}(X)[d^\perp]} \neq \emptyset$.

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Theorem (N 2017)

Let $\pi \colon X \to Z$ be a morphism between smooth projective geometrically integral varieties over k. Suppose that Z satisfies weak approximation. Suppose that the generic fiber Y is a $k(Z)$-torsor under an abelian variety $A/k(Z)$, and d be its period. If $B \subset \mathrm{Br}(X)$ is a subgroup such that $X(\mathbb{A})^B \neq \emptyset$, then $X(\mathbb{A})^{B+\mathrm{Br}(X)[d^\perp]} \neq \emptyset$.

Applications

Given an abelian variety A of dimesion ≥ 2 and a 2-covering of $f\colon Y \to A$. Let Y' be the blow up of Y along $f^{-1}(0)$. The antipodal involution ι on A induces an involution on Y'; The quotient Y'/ι is called the Kummer variety $Kum(Y)$ attached to Y

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Corollary (Skorobogatov-Zarhin, N 2017)

Let A be an abelian variety defined over a number field k. Let X be the Kummer variety attached to a 2-covering of A. If $B \subset \mathrm{Br}(X)$ is a subgroup such that $X(\mathbb{A})^B \neq \emptyset$, then $X(\mathbb{A})^{B+\mathrm{Br}(X)[2^\perp]} \neq \emptyset$.

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For simplicity, assume $B = \{1\}$. If $X(\mathbb{A})^{\mathrm{Br}(X)[d^{\perp}]} = \emptyset$, there exists as finite subgroup $H \subset \mathrm{Br}(X)[d^{\perp}]$ such that $X(\mathbb{A})^{H} = \emptyset$. Let $N = |H|$.

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S finite set of places pairing nontrivially with elements in H. Find $P \in Z(k)$ with $Q_v \in X_P(k_v)$ for all $v \in S$. D a k-rational zero cycle of degree d on X_P.

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S finite set of places pairing nontrivially with elements in H. Find $P \in Z(k)$ with $Q_v \in X_P(k_v)$ for all $v \in S$. D a k-rational zero cycle of degree d on X_P.
Define

$$C_v := nD + m(Q_v)$$

where n, m are so that C_v has degree 1 and $N \mid m$. By property (ZC) each $C_v \sim (R_v)$ for some $R_v \in X_P(k_v)$. Choose arbitrary points $R_v \in X(k_v)$ for $v \notin S$. Then $\{R_v\} \in X(\mathbb{A})^H$, a contradiction.

The End