Local to global principle for the moduli space of K3 surfaces

Gregorio Baldi

Workshop on Galois representations and K3 surfaces organised by Martin Orr and Alexei Skorobogatov

02/05/2018

Notations

In this talk we work with:

- K a number field, $\overline{K}$ a fixed algebraic closure, $\mathrm{Gal}(\overline{K}/K)$ its absolute Galois group;

Notations

In this talk we work with:

- K a number field, $\overline{K}$ a fixed algebraic closure, $\mathrm{Gal}(\overline{K}/K)$ its absolute Galois group;
- v a finite a place of K, and we write K_v for the associated local field and $\mathrm{Gal}(\overline{K}_v/K_v)$ for its absolute Galois group. We denote by Frob_v the conjugacy class of a lift of a topological generator of the absolute Galois group of the residue field at v;

Notations

In this talk we work with:

- K a number field, $\overline{K}$ a fixed algebraic closure, $\mathrm{Gal}(\overline{K}/K)$ its absolute Galois group;
- v a finite a place of K, and we write K_v for the associated local field and $\mathrm{Gal}(\overline{K}_v/K_v)$ for its absolute Galois group. We denote by Frob_v the conjugacy class of a lift of a topological generator of the absolute Galois group of the residue field at v;
- a fixed embedding $\overline{K} \hookrightarrow \mathbb{C}$.

Motivation: section conjecture for the moduli space of abelian varieties

$\mathcal{A}_{\mathrm{g}} :=$ moduli space of p.p.a.v. of dimension g;
It is a Deligne-Mumford stack (or an orbifold) defined over $\mathbb{Q}$.

Motivation: section conjecture for the moduli space of abelian varieties

$\mathcal{A}_g :=$ moduli space of p.p.a.v. of dimension g;
It is a Deligne-Mumford stack (or an orbifold) defined over $\mathbb{Q}$.

Question

Are sections s of $\mathcal{A}_g /K$ locally induced by points induced by global points?

Selmer set and family of Galois representations

When $g > 1$ we have:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(\mathcal{A}_{g,\mathbb{C}}) & \longrightarrow & \pi_1(\mathcal{A}_g) & \longrightarrow & \mathrm{Gal}(\overline{K}/K) & \to & 1 \\ & & \downarrow \cong & & \downarrow & & \downarrow \chi & & \\ 1 & \longrightarrow & \mathrm{Sp}_{2g}(\widehat{\mathbb{Z}}) & \longrightarrow & \mathrm{GSp}_{2g}(\widehat{\mathbb{Z}}) & \longrightarrow & \widehat{\mathbb{Z}}^* & \longrightarrow & 1 \end{array},$$

Selmer set and family of Galois representations

When $g > 1$ we have:

$$
\begin{array}{ccccccccc}
 & & & & & \overset{\text{sections}}{\curvearrowleft} & & & \\
1 & \longrightarrow & \pi_1(\mathcal{A}_{g,\mathbb{C}}) & \longrightarrow & \pi_1(\mathcal{A}_g) & \longrightarrow & \mathrm{Gal}(\overline{K}/K) & \to & 1 \\
 & & \downarrow \cong & & \downarrow & & \downarrow \chi & & \\
1 & \longrightarrow & \mathrm{Sp}_{2g}(\widehat{\mathbb{Z}}) & \longrightarrow & \mathrm{GSp}_{2g}(\widehat{\mathbb{Z}}) & \longrightarrow & \widehat{\mathbb{Z}}^* & \longrightarrow & 1
\end{array}.
$$

Sections $\rightsquigarrow$ ℓ-adic families of Galois representations.

Selmer set and family of Galois representations

When $g > 1$ we have:

$$\begin{array}{ccccccccc}
 & & & & & \overset{\text{sections}}{\curvearrowleft} & & & \\
1 & \longrightarrow & \pi_1(\mathcal{A}_{g,\mathbb{C}}) & \longrightarrow & \pi_1(\mathcal{A}_g) & \longrightarrow & \mathrm{Gal}(\overline{K}/K) & \to & 1 \\
 & & \downarrow{\scriptstyle \cong} & & \downarrow & & \downarrow{\scriptstyle \chi} & & \\
1 & \longrightarrow & \mathrm{Sp}_{2\mathrm{g}}(\widehat{\mathbb{Z}}) & \longrightarrow & \mathrm{GSp}_{2\mathrm{g}}(\widehat{\mathbb{Z}}) & \longrightarrow & \widehat{\mathbb{Z}}^* & \longrightarrow & 1
\end{array}.$$

Sections $\rightsquigarrow$ ℓ-adic families of Galois representations.

Question

Is it possible to find some 'local' representation-theoretical properties to ensure that a family of ℓ-adic reps comes from an abelian variety?

Weakly compatible family of ℓ-adic representations

Definition (Weakly compatible, after Serre)

A family $\{\rho_\ell : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_n(\mathbb{Q}_\ell)\}_\ell$ is *weakly compatible* if there exists a finite set of places Σ of K such that

(i) for all ℓ, ρ_ℓ is unramified outside the union of Σ and the places Σ_ℓ of K dividing ℓ;

Weakly compatible family of ℓ-adic representations

Definition (Weakly compatible, after Serre)

A family $\{\rho_\ell : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_n(\mathbb{Q}_\ell)\}_\ell$ is *weakly compatible* if there exists a finite set of places Σ of K such that

(i) for all ℓ, ρ_ℓ is unramified outside the union of Σ and the places Σ_ℓ of K dividing ℓ;

(ii) for all $v \notin \Sigma \cup \Sigma_\ell$, denoting by Frob_v a frobenius element at v, the characteristic polynomial of $\rho_\ell(\mathrm{Frob}_v)$ has rational coefficient and it is independent of ℓ.

Weakly compatible family of ℓ-adic representations

Definition (Weakly compatible, after Serre)

A family $\{\rho_\ell : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_n(\mathbb{Q}_\ell)\}_\ell$ is *weakly compatible* if there exists a finite set of places Σ of K such that

(i) for all ℓ, ρ_ℓ is unramified outside the union of Σ and the places Σ_ℓ of K dividing ℓ;

(ii) for all $v \notin \Sigma \cup \Sigma_\ell$, denoting by Frob_v a frobenius element at v, the characteristic polynomial of $\rho_\ell(\mathrm{Frob}_v)$ has rational coefficient and it is independent of ℓ.

Example (Deligne)

If X is a smooth projective variety defined over K, $\{H^i_{\text{et}}(X_{\overline{K}}, \mathbb{Q}_\ell(j))\}_\ell$ form a weakly compatible system.

Patrikis-Voloch-Zarhin's result (2016)

Let $\{\rho_\ell : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_{2N}(\mathbb{Q}_\ell)\}_\ell$ be a weakly compatible system such that for some primes ℓ_0, ℓ_1, ℓ_2 we have

- ρ_{ℓ_0} is de Rham at all places of K above ℓ_0;

Patrikis-Voloch-Zarhin's result (2016)

Let $\{\rho_\ell : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_{2N}(\mathbb{Q}_\ell)\}_\ell$ be a weakly compatible system such that for some primes ℓ_0, ℓ_1, ℓ_2 we have

- ρ_{ℓ_0} is de Rham at all places of K above ℓ_0;
- ρ_{ℓ_1} is absolutely irreducible;

Patrikis-Voloch-Zarhin's result (2016)

Let $\{\rho_\ell : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_{2N}(\mathbb{Q}_\ell)\}_\ell$ be a weakly compatible system such that for some primes ℓ_0, ℓ_1, ℓ_2 we have

- ρ_{ℓ_0} is de Rham at all places of K above ℓ_0;
- ρ_{ℓ_1} is absolutely irreducible;
- there is at least one place $v \in \Sigma_{\ell_2}$, such that $\rho_{\ell_2 | \mathrm{Gal}(\overline{K_v}, K_v)}$ is de Rham with Hodge-Tate weights $-1, 0$ each with multiplicity N.

Patrikis-Voloch-Zarhin's result (2016)

Let $\{\rho_\ell : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_{2N}(\mathbb{Q}_\ell)\}_\ell$ be a weakly compatible system such that for some primes ℓ_0, ℓ_1, ℓ_2 we have

- ρ_{ℓ_0} is de Rham at all places of K above ℓ_0;
- ρ_{ℓ_1} is absolutely irreducible;
- there is at least one place $v \in \Sigma_{\ell_2}$, such that $\rho_{\ell_2 | \mathrm{Gal}(\overline{K_v}, K_v)}$ is de Rham with Hodge-Tate weights $-1, 0$ each with multiplicity N.

Then, assuming some well known conjectures, there exists an abelian variety A defined over K such that $\rho_\ell \cong V_\ell(A)$ for all ℓ.

Formalism of motives

For any field E of characteristic zero, we denote by

$$\mathcal{M}_{K,E}$$

the category of homological motives over K with coefficients in E.

Formalism of motives

For any field E of characteristic zero, we denote by

$$\mathcal{M}_{K,E}$$

the category of homological motives over K with coefficients in E. Assuming the Tate conjecture it is a semisimple category and it is equivalent to the category $\mathrm{Rep}(\mathcal{G}_{K,E})$ for some pro-reductive group $\mathcal{G}_{K,E}$ (choosing an E-linear fibre functor).

Formalism of motives

For any field E of characteristic zero, we denote by

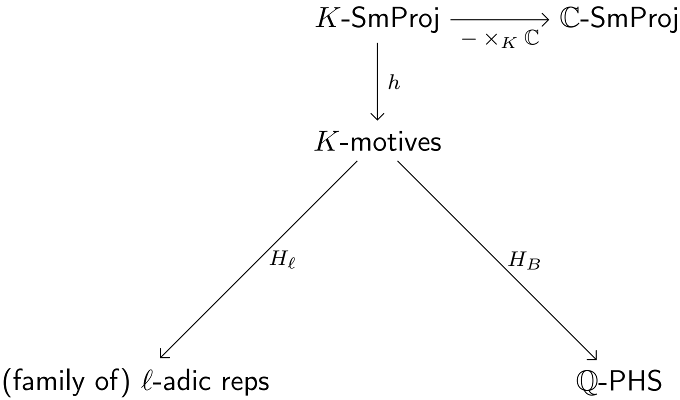
$$\mathcal{M}_{K,E}$$

the category of homological motives over K with coefficients in E. Assuming the Tate conjecture it is a semisimple category and it is equivalent to the category $\mathrm{Rep}(\mathcal{G}_{K,E})$ for some pro-reductive group $\mathcal{G}_{K,E}$ (choosing an E-linear fibre functor).
We fix a family of embeddings $\iota_\ell : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_\ell$ and write

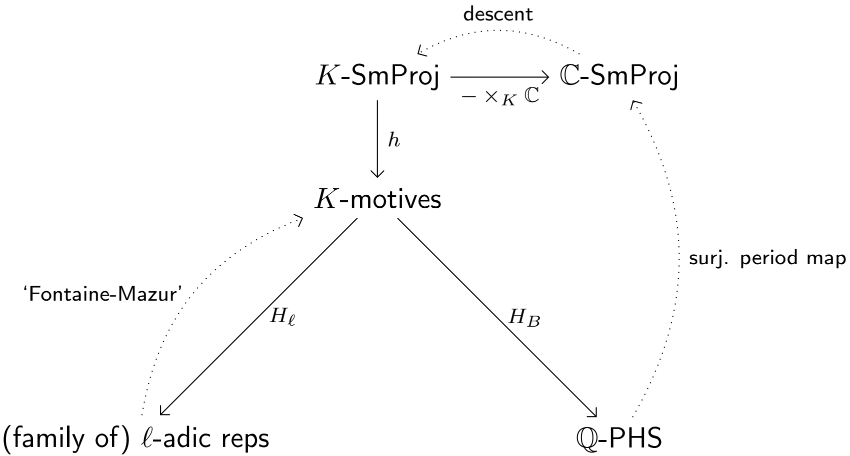
$$H_\ell : \mathcal{M}_{K,E} \to \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\mathrm{Gal}(\overline{K}/K))$$

for the ℓ-adic realisation functors associated to ι_ℓ.

A picture


K-SmProj
$\mathbb{C}$-SmProj
$- \times_K \mathbb{C}$
h
K-motives
H_ℓ
H_B
(family of) ℓ-adic reps
$\mathbb{Q}$-PHS

A picture


descent
K-SmProj
$\mathbb{C}$-SmProj
$- \times_K \mathbb{C}$
h
K-motives
surj. period map
'Fontaine-Mazur'
H_ℓ
H_B
(family of) ℓ-adic reps
$\mathbb{Q}$-PHS

The conjecture

Conjecture

Let $r_\ell : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_n(\mathbb{Q}_\ell)$ *be an irreducible geometric Galois representation. Then there exists an object* $M \in \mathcal{M}_{K,\overline{\mathbb{Q}}}$ *such that*

$$r_\ell \otimes \overline{\mathbb{Q}}_\ell \cong H_\ell(M) \in \mathit{Rep}_{\overline{\mathbb{Q}}_\ell}(\mathrm{Gal}(\overline{K}/K)).$$

The conjecture

Conjecture

Let $r_\ell : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_n(\mathbb{Q}_\ell)$ *be an irreducible geometric Galois representation. Then there exists an object* $M \in \mathcal{M}_{K,\overline{\mathbb{Q}}}$ *such that*

$$r_\ell \otimes \overline{\mathbb{Q}}_\ell \cong H_\ell(M) \in \mathit{Rep}_{\overline{\mathbb{Q}}_\ell}(\mathrm{Gal}(\overline{K}/K)).$$

Remark

We work with compatible systems of ℓ-adic reps, rather than a fixed ρ_ℓ, to produce an object in $\mathcal{M}_K$, rather than $\mathcal{M}_{K,\overline{\mathbb{Q}}}$.

Some implications

This conjectural description of essential image of the ℓ-adic realisation functor is a consequence of the following well known conjectures:

Some implications

This conjectural description of essential image of the ℓ-adic realisation functor is a consequence of the following well known conjectures:

1. Tate conjecture (i.e. H_ℓ is a full functor);

Some implications

This conjectural description of essential image of the ℓ-adic realisation functor is a consequence of the following well known conjectures:

1. Tate conjecture (i.e. H_ℓ is a full functor);
2. Serre's semisimplicity conjecture (i.e. H_ℓ takes value in semisimple Galois reps);

Some implications

This conjectural description of essential image of the ℓ-adic realisation functor is a consequence of the following well known conjectures:

1. Tate conjecture (i.e. H_ℓ is a full functor);
2. Serre's semisimplicity conjecture (i.e. H_ℓ takes value in semisimple Galois reps);
3. Fontaine-Mazur (an irreducible ℓ-adic Galois representation is *geometric* iff it comes from geometry).

Some implications

This conjectural description of essential image of the ℓ-adic realisation functor is a consequence of the following well known conjectures:

1. Tate conjecture (i.e. H_ℓ is a full functor);
2. Serre's semisimplicity conjecture (i.e. H_ℓ takes value in semisimple Galois reps);
3. Fontaine-Mazur (an irreducible ℓ-adic Galois representation is *geometric* iff it comes from geometry).

Theorem (B. Moonen (2017))

The Tate conjecture implies the semisimplicity conjecture.

Some implications

This conjectural description of essential image of the ℓ-adic realisation functor is a consequence of the following well known conjectures:

1. Tate conjecture (i.e. H_ℓ is a full functor);
2. Serre's semisimplicity conjecture (i.e. H_ℓ takes value in semisimple Galois reps);
3. Fontaine-Mazur (an irreducible ℓ-adic Galois representation is *geometric* iff it comes from geometry).

Theorem (B. Moonen (2017))

The Tate conjecture implies the semisimplicity conjecture.

K3 surfaces and Galois representations

Question

Given a weakly compatible system of ℓ-adic representations that looks like the cohomology of a K3 surface, is it induced by a K3 surface?

Question

Given a weakly compatible system of ℓ-adic representations that looks like the cohomology of a K3 surface, is it induced by a K3 surface?

Irreducibility issue: it is better to work with the transcendental part (i.e the orthogonal compliment of the Neron-Severi in the H^2).

K3 surfaces and Galois representations

Question

Given a weakly compatible system of ℓ-adic representations that looks like the cohomology of a K3 surface, is it induced by a K3 surface?

Irreducibility issue: it is better to work with the transcendental part (i.e the orthogonal compliment of the Neron-Severi in the H^2).

Question

Given $\{\rho_\ell\}_\ell$ a weakly compatible system of ℓ-adic representations of $\mathrm{Gal}(\overline{K}/K)$ that looks like the transcendental part of a K3 surface, can we construct a K3 surface X (defined over K) such that $T(X_{\overline{K}})_{\mathbb{Q}_\ell} \cong \rho_\ell$ for all ℓs?

Motive of a surface (after Murre-Pedrini)

We can isolate the transcendental part of the motive of a surface X:

$$h_2(X) = \left(h^2_{alg}(X) \oplus t_2(X)\right),$$

where $h^2_{alg}(X) = (X, \pi_2^{alg}, 0)$ and $t_2(X) = (X, \pi_2^{tr}, 0)$, for a refined Künneth decomposition $\pi_2 = \pi_2^{alg} + \pi_2^{tr}$.

Motive of a surface (after Murre-Pedrini)

We can isolate the transcendental part of the motive of a surface X:

$$h_2(X) = \left(h^2_{alg}(X) \oplus t_2(X)\right),$$

where $h^2_{alg}(X) = (X, \pi_2^{alg}, 0)$ and $t_2(X) = (X, \pi_2^{tr}, 0)$, for a refined Künneth decomposition $\pi_2 = \pi_2^{alg} + \pi_2^{tr}$. We have

$$H_B(h^2_{alg}(X) \oplus t_2(X)) = \mathrm{NS}(X)_{\mathbb{Q}} \oplus T(X)_{\mathbb{Q}},$$

$$H_\ell(h^2_{alg}(X) \oplus t_2(X)) = \mathrm{NS}(X_{\overline{K}})_{\mathbb{Q}_\ell} \oplus T(X_{\overline{K}})_{\mathbb{Q}_\ell}.$$

Local conditions

For a *refined Fontaine-Mazur* we need to work with the following local conditions:

1. For some prime ℓ_0, ρ_{ℓ_0} is de Rham at all places of K above ℓ_0;

Local conditions

For a *refined Fontaine-Mazur* we need to work with the following local conditions:

1. For some prime ℓ_0, ρ_{ℓ_0} is de Rham at all places of K above ℓ_0;
2. For some prime ℓ_1, ρ_{ℓ_1} is absolutely irreducible;

Local conditions

For a *refined Fontaine-Mazur* we need to work with the following local conditions:

1. For some prime ℓ_0, ρ_{ℓ_0} is de Rham at all places of K above ℓ_0;
2. For some prime ℓ_1, ρ_{ℓ_1} is absolutely irreducible;
3. For some prime ℓ_2 and at least one place $v \in \Sigma_{\ell_2}$, $\rho_{\ell_2 | \mathrm{Gal}(\overline{K}_v/K_v)}$ is de Rham with Hodge-Tate weights of a K3 surface, and multiplicities, respectively, $1, 20-\rho, 1$.

Local conditions

For a *refined Fontaine-Mazur* we need to work with the following local conditions:

1. For some prime ℓ_0, ρ_{ℓ_0} is de Rham at all places of K above ℓ_0;
2. For some prime ℓ_1, ρ_{ℓ_1} is absolutely irreducible;
3. For some prime ℓ_2 and at least one place $v \in \Sigma_{\ell_2}$, $\rho_{\ell_2 | \mathrm{Gal}(\overline{K}_v/K_v)}$ is de Rham with Hodge-Tate weights of a K3 surface, and multiplicities, respectively, $1, 20-\rho, 1$.

Note that condition (3) is satisfied if there exists a $K3$ surface X_v/K_v of Picard rank ρ and $\rho_{\ell_2 | \mathrm{Gal}(\overline{K}_v/K_v)}$ is isomorphic to the representation induced by $T(X_{\overline{K_v}})_{\mathbb{Q}_\ell}$.

Main theorem

Theorem

Let $\rho \in \mathbb{N}$ *be such that* $2 < 22 - \rho \leq 10$. *Assume the Tate, Fontaine-Mazur and the Hodge conjecture. Let*

$$\{\rho_\ell : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_{22-\rho}(\mathbb{Q}_\ell)\}_\ell$$

be a weakly compatible family of ℓ*-adic representations satisfying the conditions (1), (2), (3).*

Main theorem

Theorem

Let $\rho \in \mathbb{N}$ be such that $2 < 22 - \rho \leq 10$. Assume the Tate, Fontaine-Mazur and the Hodge conjecture. Let

$$\{\rho_\ell : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_{22-\rho}(\mathbb{Q}_\ell)\}_\ell$$

be a weakly compatible family of ℓ-adic representations satisfying the conditions (1), (2), (3).
Then there exists a simple motive M defined over K inducing the representations ρ_ℓ

Main theorem

Theorem

Let $\rho \in \mathbb{N}$ *be such that* $2 < 22 - \rho \leq 10$. *Assume the Tate, Fontaine-Mazur and the Hodge conjecture. Let*

$$\{\rho_\ell : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_{22-\rho}(\mathbb{Q}_\ell)\}_\ell$$

be a weakly compatible family of ℓ*-adic representations satisfying the conditions (1), (2), (3).*
Then there exists a simple motive M *defined over* K *inducing the representations* ρ_ℓ *and a finite extension* L/K*, such that the base change of* M *to* L *is isomorphic to the transcendental part of the motive of a K3 surface defined over* L.

Strategy of the proof

- From $\{\rho_\ell\}$ construct a motive M defined over K inducing $\{\rho_\ell\}_\ell$ and giving a Hodge structure of K3 type;

Strategy of the proof

- From $\{\rho_\ell\}$ construct a motive M defined over K inducing $\{\rho_\ell\}_\ell$ and giving a Hodge structure of K3 type;
- Construct a complex K3 X from that Hodge structure;

Strategy of the proof

- From $\{\rho_\ell\}$ construct a motive M defined over K inducing $\{\rho_\ell\}_\ell$ and giving a Hodge structure of K3 type;
- Construct a complex K3 X from that Hodge structure;
- Descend the K3 to a number field (using the fact that $t_2(X) \cong M_{|\mathbb{C}}$).

Strategy of the proof

- From $\{\rho_\ell\}$ construct a motive M defined over K inducing $\{\rho_\ell\}_\ell$ and giving a Hodge structure of K3 type;
- Construct a complex K3 X from that Hodge structure;
- Descend the K3 to a number field (using the fact that $t_2(X) \cong M_{|\mathbb{C}}$).

Two problems:

- We do not have enough information to reconstruct the algebraic part of the H^2. This is why we need a finite extension...

Strategy of the proof

- From $\{\rho_\ell\}$ construct a motive M defined over K inducing $\{\rho_\ell\}_\ell$ and giving a Hodge structure of K3 type;
- Construct a complex K3 X from that Hodge structure;
- Descend the K3 to a number field (using the fact that $t_2(X) \cong M_{|\mathbb{C}}$).

Two problems:

- We do not have enough information to reconstruct the algebraic part of the H^2. This is why we need a finite extension. . .
- the transcendental part determines the full H^2 only in particular cases (Nikulin). . .

Proof

Choosing a place ℓ_0 as in (1), our conjectural description of the essential image of H_{ℓ_0} ensures the existence of a motivic Galois representation

$$\rho : \mathcal{G}_{K,E} \to \mathsf{GL}_{22-\rho,E}$$

for some number field E, such that $H_{\ell_0}(\rho) \cong \rho_{\ell_0} \otimes \overline{\mathbb{Q}}_{\ell_0}$ (the same holds for every ℓ).

Proof

Choosing a place ℓ_0 as in (1), our conjectural description of the essential image of H_{ℓ_0} ensures the existence of a motivic Galois representation

$$\rho : \mathcal{G}_{K,E} \to \mathsf{GL}_{22-\rho,E}$$

for some number field E, such that $H_{\ell_0}(\rho) \cong \rho_{\ell_0} \otimes \overline{\mathbb{Q}}_{\ell_0}$ (the same holds for every ℓ). The obstruction to descending ρ to a $\mathbb{Q}$-rational representation of $\mathcal{G}_K$ is an element $\mathsf{obs}_\rho \in H^1(\mathrm{Gal}(E/\mathbb{Q}), \mathrm{PGL}_{22-\rho}(E))$.

Lemma (P-V-Z)

In fact obs$_\rho$ lies in

$$\ker(H^1(\mathrm{Gal}(E/\mathbb{Q}), \mathrm{PGL}_{22-\rho}(E)) \to \prod_{\ell}(\mathrm{Gal}(E_\lambda/\mathbb{Q}_\lambda), \mathrm{PGL}_{22-\rho}(E_\lambda))).$$

This shows that the compatible system $\{\rho_\ell\}_\ell$ arises as the ℓ-adic realisations of a motive $M \in \mathcal{M}_K$ of rank $22 - \rho$.

This shows that the compatible system $\{\rho_\ell\}_\ell$ arises as the ℓ-adic realisations of a motive $M \in \mathcal{M}_K$ of rank $22 - \rho$. Since ρ_{ℓ_1} is absolutely irreducible, M is absolutely simple.

This shows that the compatible system $\{\rho_\ell\}_\ell$ arises as the ℓ-adic realisations of a motive $M \in \mathcal{M}_K$ of rank $22-\rho$. Since ρ_{ℓ_1} is absolutely irreducible, M is absolutely simple. Working at the prime ℓ_2, where $\rho_{\ell_2 | \mathrm{Gal}(\overline{K}_v/K_v)}$ is de Rham with Hodge-Tate weights of a K3 surface, we have a comparison isomorphism

$$H_{dR}(M) \otimes_K K_v \cong D_{dR,K_v}(H_{\ell_2}(M)).$$

This shows that the compatible system $\{\rho_\ell\}_\ell$ arises as the ℓ-adic realisations of a motive $M \in \mathcal{M}_K$ of rank $22-\rho$. Since ρ_{ℓ_1} is absolutely irreducible, M is absolutely simple. Working at the prime ℓ_2, where $\rho_{\ell_2 | \mathrm{Gal}(\overline{K}_v/K_v)}$ is de Rham with Hodge-Tate weights of a K3 surface, we have a comparison isomorphism

$$H_{dR}(M) \otimes_K K_v \cong D_{dR,K_v}(H_{\ell_2}(M)).$$

Tanks to the Betti-de Rham comparison isomorphism we conclude that $H_B(M_{|\mathbb{C}})$ is a polarizable rational Hodge structure of weight two and with Hodge numbers $1-(20-\rho)-1$, since $\rho_{\ell_2 | \mathrm{Gal}(\overline{K}_v/K_v)}$ has such multiplicities.

Surjectivity of the period map

We may apply the following proposition to obtain a a K3 surface $X/\mathbb{C}$ with transcendental part isomorphic to $H_B(M_{|\mathbb{C}})$.

Surjectivity of the period map

We may apply the following proposition to obtain a a K3 surface $X/\mathbb{C}$ with transcendental part isomorphic to $H_B(M_{|\mathbb{C}})$.

Proposition (van Geemen)

Let (V, h, ψ) be a $\mathbb{Q}$-PHS of K3 type with $\mathrm{End}_{Hdg}(V) = \mathbb{Q}$, and

$$3 \leq \dim V \leq 10$$

Choose a free $\mathbb{Z}$-module $T \subset V$, compatibly with the Hodge structure, of rank $\dim_{\mathbb{Q}} V$ such that ψ is integer valued on $T \times T$. Then there exists a K3 surface $X/\mathbb{C}$ with $T(X) \cong T$ as integral polarised Hodge structure.

Surjectivity of the period map

We may apply the following proposition to obtain a a K3 surface $X/\mathbb{C}$ with transcendental part isomorphic to $H_B(M_{|\mathbb{C}})$.

Proposition (van Geemen)

Let (V, h, ψ) be a $\mathbb{Q}$-PHS of K3 type with $\mathrm{End}_{Hdg}(V) = \mathbb{Q}$, and

$$3 \leq \dim V \leq 10$$

Choose a free $\mathbb{Z}$-module $T \subset V$, compatibly with the Hodge structure, of rank $\dim_{\mathbb{Q}} V$ such that ψ is integer valued on $T \times T$. Then there exists a K3 surface $X/\mathbb{C}$ with $T(X) \cong T$ as integral polarised Hodge structure.

Under these numerical constraints, a theorem Nikulin shows that there exists a primitive embedding of lattices

$$T \hookrightarrow \Lambda_{K3}.$$

We have constructed a K3 $X/\mathbb{C}$ such that

$$T(X)_{\mathbb{Q}} \cong H_B(M_{|\mathbb{C}}).$$

We have constructed a K3 $X/\mathbb{C}$ such that

$$T(X)_{\mathbb{Q}} \cong H_B(M_{|\mathbb{C}}).$$

Thanks to the Hodge conjecture we can lift the isomorphism of Hodge structures to get an isomorphism at the level of motives:

$$t_2(X) \cong M_{|\mathbb{C}} \in \mathcal{M}_{\mathbb{C}},$$

where $t_2(X)$ is the transcendental part of the motive of X

Since M is defined over a number field, for all $\sigma \in \mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})$, we have the following chain of isomorphisms:

$$^{\sigma}t_2(X) \cong {}^{\sigma}M_{|\mathbb{C}} = M_{|\mathbb{C}} \cong t_2(X) \in \mathcal{M}_{\mathbb{C}}.$$

Since M is defined over a number field, for all $\sigma \in \operatorname{Aut}(\mathbb{C}/\overline{\mathbb{Q}})$, we have the following chain of isomorphisms:

$$^{\sigma}t_2(X) \cong {}^{\sigma}M_{|\mathbb{C}} = M_{|\mathbb{C}} \cong t_2(X) \in \mathcal{M}_{\mathbb{C}}.$$

It follows that, for all$\sigma \in \operatorname{Aut}(\mathbb{C}/\overline{\mathbb{Q}})$ we have an isomorphism of $\mathbb{Q}$-PHS

$$T(X)_{\mathbb{Q}} \cong T({}^{\sigma}X)_{\mathbb{Q}}.$$

Descent

We are left to prove the following.

Theorem

Let $X/\mathbb{C}$ be a K3 surface such that for all $\sigma \in \mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})$ we have an isomorphism of $\mathbb{Q}$-PHS

$$T(X)_{\mathbb{Q}} \cong T({}^{\sigma}X)_{\mathbb{Q}}.$$

Then X admits a model defined over $\overline{\mathbb{Q}}$.

The descent theorem then follows from the following two remarks:

The descent theorem then follows from the following two remarks:

- The number of complex K3 surfaces, up to isomorphism, Y such that $T(Y)_{\mathbb{Q}}$ is isomorphic to $T(X)_{\mathbb{Q}}$ is at most countable;

The descent theorem then follows from the following two remarks:

- The number of complex K3 surfaces, up to isomorphism, Y such that $T(Y)_{\mathbb{Q}}$ is isomorphic to $T(X)_{\mathbb{Q}}$ is at most countable;
- If all the conjugates of X fall into countably many isomorphism classes, then X descends to a number field.

The descent theorem then follows from the following two remarks:

- The number of complex K3 surfaces, up to isomorphism, Y such that $T(Y)_{\mathbb{Q}}$ is isomorphic to $T(X)_{\mathbb{Q}}$ is at most countable;
- If all the conjugates of X fall into countably many isomorphism classes, then X descends to a number field.

Proof.

For the first point, use the fact there X admits only finitely many Fourier-Mukai partners (Mukai).

The descent theorem then follows from the following two remarks:

- The number of complex K3 surfaces, up to isomorphism, Y such that $T(Y)_{\mathbb{Q}}$ is isomorphic to $T(X)_{\mathbb{Q}}$ is at most countable;
- If all the conjugates of X fall into countably many isomorphism classes, then X descends to a number field.

Proof.

For the first point, use the fact there X admits only finitely many Fourier-Mukai partners (Mukai).
For the second, use that K3s (with some extra structure) have a fine moduli space defined over $\overline{\mathbb{Q}}$ (Rizov). □

How to get rid of the extension L/K?

Question

Assume that $\mathcal{M}_K$ is a semisimple neutral Tannakian category over $\mathbb{Q}$. Let $M \in \mathcal{M}_K$ be a simple motive defined over some number field K. Assume there exists a finite extension L/K such that M_L is isomorphic to the transcendental part of the motive of Y_L, a K3 surface defined over L. Is there a K3 surface X defined over K such that

$$t_2(X) \cong M \in \mathcal{M}_K.$$

Answer for abelian varieties

Proposition

Let K be a number field, and assume that the category $\mathcal{M}_K$ is a semisimple neutral Tannakian category over $\mathbb{Q}$. Let $M \in \mathcal{M}_K$ be a simple motive such that, after a finite extension L/K,

$$M_L \cong H_1(A_L) \in \mathcal{M}_L$$

for some abelian variety A_L defined over L.

Answer for abelian varieties

Proposition

Let K be a number field, and assume that the category $\mathcal{M}_K$ is a semisimple neutral Tannakian category over $\mathbb{Q}$. Let $M \in \mathcal{M}_K$ be a simple motive such that, after a finite extension L/K,

$$M_L \cong H_1(A_L) \in \mathcal{M}_L$$

for some abelian variety A_L defined over L.
Then there exists an abelian variety A/K such that

$$M \cong H_1(A) \in \mathcal{M}_K.$$

Proof

Faltings proved that the following functor is full (and faithful):

$$H_1(-) : \mathsf{AV}_K^0 \to \mathcal{M}_K, \quad B \mapsto H_1(B).$$

Consider the K-ab. var. $\mathrm{Res}_{L,K}(A_L)$ and notice that $H_1(\mathrm{Res}_{L,K}(A_L))$ corresponds to $\mathsf{Ind}_L^K(H_1(A_L))$.

Proof

Faltings proved that the following functor is full (and faithful):

$$H_1(-) : \mathsf{AV}_K^0 \to \mathcal{M}_K, \quad B \mapsto H_1(B).$$

Consider the K-ab. var. $\mathrm{Res}_{L,K}(A_L)$ and notice that $H_1(\mathrm{Res}_{L,K}(A_L))$ corresponds to $\mathsf{Ind}_L^K(H_1(A_L))$. Since $M_L \cong H_1(A_L)$, Frobenius reciprocity implies that

$$\mathrm{Hom}_{\mathcal{M}_K}(M, \mathsf{Ind}_L^K(H_1(A_L))) \neq 0.$$

Since M is simple, an element in such Hom-set realizes M as a direct summand of $H_1(\mathrm{Res}_{L,K}(A_L))$ in $\mathcal{M}_K$, therefore in AV_K^0.

Proof

Faltings proved that the following functor is full (and faithful):

$$H_1(-) : \mathsf{AV}_K^0 \to \mathcal{M}_K, \quad B \mapsto H_1(B).$$

Consider the K-ab. var. $\mathrm{Res}_{L,K}(A_L)$ and notice that $H_1(\mathrm{Res}_{L,K}(A_L))$ corresponds to $\mathsf{Ind}_L^K(H_1(A_L))$. Since $M_L \cong H_1(A_L)$, Frobenius reciprocity implies that

$$\mathrm{Hom}_{\mathcal{M}_K}(M, \mathsf{Ind}_L^K(H_1(A_L))) \neq 0.$$

Since M is simple, an element in such Hom-set realizes M as a direct summand of $H_1(\mathrm{Res}_{L,K}(A_L))$ in $\mathcal{M}_K$, therefore in AV_K^0.
Otherwise stated there exists an endomorphism of $\mathrm{Res}_{L,K}(A_L)$ whose image is an abelian variety A/K such that $H_1(A) \cong M \in \mathcal{M}_K$.

THANKS FOR YOUR ATTENTION!