Lie algebras
Problem Sheet 3.

1. Let $\mathfrak{g}$ be the Lie algebra of affine transformations of the line. Compute the Gram matrix of the Killing form of $\mathfrak{g}$. Check that the kernel of the Killing form of $\mathfrak{g}$ is $\mathfrak{g}'$, so that the Killing form is not identically zero.

2. Prove that the quotient of a Lie algebra $\mathfrak{g}$ by its radical is semisimple.

3. Prove that any simple Lie algebra is semisimple.

4. Prove directly from the defintion that $\mathfrak{sl}(2)$ is a simple Lie algebra. (Hint: Let E_{ij}, $i \neq j$, be the matrix with the ij-entry equal to 1, and all the other entries equal to 0. If $\mathfrak{a} \subset \mathfrak{sl}(2)$ is a non-zero ideal, and $a \in \mathfrak{a}$, $a \neq 0$, then $[E_{ij}, a] \in \mathfrak{a}$. Use this to prove that $E_{mn} \in \mathfrak{a}$ for some m and n. Deduce that $\mathfrak{a} = \mathfrak{sl}(2)$.)

5. Let $\mathfrak{g}$ be a Lie algebra, and $\mathfrak{a} \subset \mathfrak{g}$ a semisimple ideal. Prove that there exists an ideal $\mathfrak{b} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ is a direct sum of Lie algebras.

6. Let $\mathfrak{g}$ be a Lie algebra of dimension n, and $x \in \mathfrak{g}$. Prove that $\mathrm{ad}(x)$ is nilpotent if and only if the leading term equal to 1 is the only non-zero coefficient of the characteristic polynomial of $\mathrm{ad}(x)$. (Hint: this is a general fact about nilpotent linear transformations.)

7. Prove that the constant term of the characteristic polynomial of $\mathrm{ad}(x)$ for any $x \in \mathfrak{g}$ is identically zero.