Lie algebras
Problem Sheet 2.

1. Let $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ be the adjoint representation of $\mathfrak{g}$. Show that

$$\mathrm{ad}(\mathfrak{g}') \subset \mathfrak{sl}(\mathfrak{g}).$$

2. Let $Z(\mathfrak{g}) = \{x \in \mathfrak{g} | [xy] = 0 \text{ for any } y \in \mathfrak{g}\}$ be the centre of $\mathfrak{g}$.

(a) Prove that if $\mathfrak{g}$ is a non-abelian Lie algebra, then

$$\dim(Z(\mathfrak{g})) \leq \dim(\mathfrak{g}) - 2.$$

(b) Give an example of a Lie algebra $\mathfrak{g}$ such that $\dim(Z(\mathfrak{g})) = \dim(\mathfrak{g}) - 2$ and $Z(\mathfrak{g}) \neq 0$.

3. Let $f : \mathfrak{g}_1 \to \mathfrak{g}_2$ be a surjective Lie algebra homomorphism. Which of the following statements are true? (Prove or give a counter-example.)

(a) $f(\mathfrak{g}_1') = \mathfrak{g}_2'$

(b) $Z(\mathfrak{g}_1) = Z(\mathfrak{g}_2)$

4. Let $S \in \mathfrak{gl}(n)$.

(a) Prove that the elements $A \in \mathfrak{gl}(n)$ such that $AS + SA^t = 0$ form a Lie subalgebra of $\mathfrak{gl}(n)$. Call it $\mathfrak{gl}_S(n)$.

(b) Identify the Lie algebra $\mathfrak{gl}_S(n)$ when $S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

(c) Does there exist an $S \in \mathfrak{gl}(n)$ such that $\mathfrak{gl}_S(n) = \mathfrak{gl}(n)$?

(d) Show that if P is an invertible $n \times n$ matrix, then $\mathfrak{gl}_S(n)$ and $\mathfrak{gl}_{PSP^t}(n)$ are isomorphic Lie algebras.

5. (a) Prove that a Lie algebra is solvable if and only if $\mathrm{ad}(\mathfrak{g})$ is solvable.

(b) Prove that a Lie algebra is nipotent if and only if $\mathrm{ad}(\mathfrak{g})$ is nilpotent.