# Lie algebras
# Problem Sheet 1.

1. (easy) Prove that $\mathfrak{o}(2)$ and $\mathfrak{n}(2)$ are abelian 1-dimensional Lie algebras, hence they are isomorphic to the ground field $k$ equipped with zero bracket.

2. (easy) Prove that the vector product defines the Lie algebra structure on $\mathbb{R}^3$. Show that this algebra is isomorphic to $\mathfrak{o}(3)$ by comparing the structure constants.

3. Let $k = \mathbb{R}$ or $\mathbb{C}$. The Lie algebras $\mathfrak{sl}(2)$, $\mathfrak{o}(3)$, $\mathfrak{t}(2)$, $\mathfrak{n}(3)$ all have dimension 3. Are any of these isomorphic? (The answer will depend on $k$.)

4. Compute the derived series of $\mathfrak{t}(n)$, $\mathfrak{n}(n)$, $\mathfrak{sl}(n)$. Hence determine which of these Lie algebras are solvable.

5. Compute the lower central series of $\mathfrak{t}(n)$, $\mathfrak{n}(n)$, $\mathfrak{sl}(n)$. Hence determine which of these Lie algebras are nilpotent.

6. Prove that $\mathfrak{g}^{(r)} \subset \mathfrak{g}^r$ for any $\mathfrak{g}$ and $r$. In particular, this implies that every nilpotent algebra is solvable. Show by an example that the converse is false. (Hint: consider the Lie algebra attached to the associative algebra of affine transformations of the line $x \mapsto ax + b$, $a$, $b \in k$. This algebra can be realized as the algebra of $2 \times 2$-matrices with the zero bottom row under the usual Lie bracket of matrices $[AB] = AB - BA$.)

7. Prove that every subalgebra of a solvable (resp. nilpotent) Lie algebra is solvable (resp. nilpotent). The same statement for quotient algebras.

8. Clearly, every element of $\mathfrak{n}(n)$ is nilpotent. Find a nilpotent subalgebra $\mathfrak{g} \subset \mathfrak{gl}(n)$ such that no non-zero element of $\mathfrak{g}$ is nilpotent. (Hint: think of diagonal matrices.)

9. Deduce from Lie's theorem that any complex (resp. real) irreducible representation of a solvable Lie algebra has dimension 1 (resp. 1 or 2).