UNIVERSITY OF LONDON

IMPERIAL COLLEGE LONDON

Course:	M4P46 MSP66 (SOLUTIONS)
Setter:	Skorobogatov
Checker:	Nikolov
Editor:	Ivanov
External:	Cremona
Date:	May 10, 2017

BSc and MSci EXAMINATIONS (MATHEMATICS)
MAY–JUNE 2009

This paper is also taken for the relevant examination for the Associateship.

M4P46 MSP66 (SOLUTIONS) Lie algebras

DATE: examdate TIME: examtime

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Setter's signature ....

Checker's signature ....

1. *i)* **4 marks, seen**

Define the derived series of $\mathfrak{g}$ inductively: $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{g}^{(n+1)} = [\mathfrak{g}^{(n)}, \mathfrak{g}^{(n)}]$. The Lie algebra $\mathfrak{g}$ is solvable if $\mathfrak{g}^{(n)} = 0$ for some n.

ii) **8 marks, seen similar**

Let $\mathfrak{t} \subset \mathfrak{sl}(2)$ be the subalgebra of upper-triangular matrices, and let $\mathfrak{n} \subset \mathfrak{sl}(2)$ be the subalgebra of strictly upper-triangular matrices. Then $[\mathfrak{t}, \mathfrak{t}] = \mathfrak{n}$ and $[\mathfrak{n}, \mathfrak{n}] = 0$, so that $\mathfrak{t}$ is solvable. But $[\mathfrak{t}, \mathfrak{n}] = \mathfrak{n}$, thus the lower central series of $\mathfrak{g}$ stabilizes at $\mathfrak{n}$ and never comes to 0. Hence $\mathfrak{t}$ is not nilpotent.

iii) **8 marks, seen similar**

Let E_{ij} be the elementary matrix, i.e. the matrix with the (i,j)-entry 1, and all the other entries 0. Then $[E_{ik}, E_{kj}] = E_{ij}$ if $i \neq j$, and $[E_{ij}, E_{ji}] = E_{ii} - E_{jj}$, hence $[\mathfrak{gl}(n), \mathfrak{gl}(n)] = \mathfrak{sl}(n)$ and $[\mathfrak{sl}(n), \mathfrak{sl}(n)] = \mathfrak{sl}(n)$. Thus the derived series of $\mathfrak{gl}(n)$ stabilizes at $\mathfrak{sl}(n)$ and never comes to 0 unless $n = 1$. Hence $\mathfrak{gl}(n)$ is not solvable for $n > 1$.

2. *i)* **5 marks, seen**

For every Lie subalgebra $\mathfrak{g} \subset \mathfrak{gl}(V)$ whose elements are nilpotent linear transformations, there exists a basis of V such that every element of $\mathfrak{g}$ is given by a strictly upper-triangular matrix. In particular, $\mathfrak{g}$ is nilpotent.

ii) **10 marks, seen similar**

$\mathfrak{g}$ has a basis of elementary matrices $A = E_{11}$, $B = E_{12}$, $C = E_{13}$. Then $[A, B] = B$, $[A, C] = C$, $[B, C] = 0$, in particular, $\mathfrak{g}$ is a Lie subalgebra. In this basis, the adjoint representation is given by

$$ad(A) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; ad(B) = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; ad(C) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Hence the matrix of the Killing form is

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

iii) **5 marks, seen similar**

The relations among A, B and C show that $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is spanned by B and C, and $[\mathfrak{g}', \mathfrak{g}'] = 0$. Thus $\mathfrak{g}'$ is solvable, and so $K(\mathfrak{g}, \mathfrak{g}') = 0$ by Cartan's first criterion. Therefore, all the entries of the matrix of the Killing form of $\mathfrak{g}$ with the exception of the $(1, 1)$-entry, are equal to 0.

3. *i)* **4 marks, seen**

Let $R \subset V$ be a root system in a real vector space V. A subset $S \subset R$ is a basis of R if S is a basis of V, and any element of R is an integral linear combination of elements of S with coefficients of the same sign.

ii) **8 marks, seen**

Let $\phi : V \to \mathbb{R}$ be a linear function taking non-zero values on the roots. Let R_+ be the set of $v \in R$ such that $\phi(v) > 0$. Let $S \subset R_+$ consist of the roots that cannot be written as $v_1 + v_2$, where v_1, $v_2 \in R_+$. Then it is clear that any element of R is an integral linear combination of elements of S with coefficients of the same sign. In particular, S spans V. If α, $\beta \in S$, then the angle between α and β is right or obtuse. (Otherwise $\alpha - \beta \in R$, and this contradicts the way S has been constructed.) Any system of vectors with such a property is linearly independent.

iii) **8 marks, seen similar**

$G_2 \subset \mathbb{C} = \mathbb{R}^2$ consists of the 6th roots of 1 in the complex plane, together with all non-zero sums of two of these vectors. Thus $|G_2| = 12$. The vectors $e_1 = 1$ and $e_2 = (-3 + \sqrt{-3})/2$ form a basis (easy check). The dimension of the semisimple Lie algebra of type G_2 is $rk(G_2) + |G_2| = 14$. The Weyl group is the dihedral group of 12 elements (all symmetries in it come from reflections in the roots).

4. *i)* **4 marks, seen**

A_3, with basis $e_1 - e_2$, $e_2 - e_3$, $e_3 - e_4$; B_3, with basis $e_1 - e_2$, $e_2 - e_3$, e_3; C_3, with basis $e_1 - e_2$, $e_2 - e_3$, $2e_3$ (note that $D_3 \simeq A_3$).

ii) **4 marks, seen**

If $\alpha_1, \ldots, \alpha_n$ is a basis of a root system R, then the Cartan matrix of R is the $n \times n$-matrix whose (i, j)-entry is $n_{\alpha\beta} = 2(\alpha, \beta)/(\beta, \beta)$.

iii) **4 marks, seen**

By (ii) the Cartan matrices of A_3, B_3, C_3, respectively, are

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}.$$

iv) **4 marks, seen**

This is the set $2v/(v,v)$, where $v \in R$.

v) **4 marks, seen**

A_3 is self-dual, and B_3 and C_3 are dual to each other.