UNIVERSITY OF LONDON

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BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2008

M4P46 MSP66

Lie algebras

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BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2008

This paper is also taken for the relevant examination for the Associateship.

M4P46 MSP66

Lie algebras

Date: examdate Time: examtime

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

In this paper all Lie algebras are defined over the field of complex numbers.

1. (i) Give the definition of a solvable Lie algebra.

   (ii) Give an example of a solvable Lie algebra which is not nilpotent.

   (iii) Prove that the matrix algebra $\mathfrak{gl}(n)$ is not solvable if $n \geq 2$.

   In parts (ii) and (iii) you are asked to justify your answers.

2. (i) State Engel's theorem. (No proof is required.)

   (ii) Let $\mathfrak{g} \subset \mathfrak{gl}(3)$ be the set of 3×3-matrices such that all the entries in the second and third rows are zero. Prove that $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(3)$. Working from the definitions, compute the matrix of the Killing form of $\mathfrak{g}$.

   (iii) Explain how your answer to (ii) illustrates Cartan's first criterion.

3. (i) Give the definition of a basis of a root system.

   (ii) Prove that every root system has a basis. (A few sentences will suffice; you are not asked to give full details of your argument.)

   (iii) Give the definition of the root system G_2. Find a basis of this root system. How many roots does G_2 have? What is the dimension of the semisimple Lie algebra of type G_2? Determine the number of elements in the Weyl group $W(G_2)$. Which well known group is $W(G_2)$ isomorphic to? (Justify your answers.)

4. Let $\mathfrak{g}$ be a semisimple Lie algebra, and let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra. Write $\mathfrak{g} = \mathfrak{g}_0 \oplus (\oplus_{\alpha \in R} \mathfrak{g}_\alpha)$, where $\mathfrak{g}_0 = \mathfrak{h}$, $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} | [h, x] = \alpha(h)x \text{ for any } h \in \mathfrak{h}\}$ and R is a finite set of non-zero linear functions $\mathfrak{h} \to \mathbb{C}$. In this question α and β are elements of $R \cup \{0\}$.

   (i) Prove that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.

   (ii) Prove that if $x \in \mathfrak{g}_\alpha$, $\alpha \neq 0$, then $ad(x) \in \mathfrak{gl}(\mathfrak{g})$ is a nilpotent linear transformation.

   (iii) Prove that if $\alpha + \beta \neq 0$, then $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are orthogonal with respect to the Killing form of $\mathfrak{g}$.

   (iv) Deduce that $R = -R$, where $-R = \{-\alpha | \alpha \in R\}$.

5. (i) List all inequivalent irreducible root systems of rank 3 by writing their simple roots and the scalar products between them. (No proof is required.)

   (ii) Give the definition of the Cartan matrix of a root system.

   (iii) Find the Cartan matrices of the root systems of rank 3. (Justify your answer.)

   (iv) Give the definition of a dual root system.

   (v) What is the dual of each of rank 3 root systems in (i)? (Justify your answer.)