Lie algebras

Test 2

10 March 2017

In this test the ground field is the field of complex numbers $\mathbb{C}$.

1. Using the results we proved in lectures, show that there is no semisimple Lie algebra which is a vector space of dimension 4.

2. Let $\mathfrak{sl}(2) \to \mathfrak{gl}(V)$ be a representation of the Lie algebra $\mathfrak{sl}(2)$ in a vector space V. Define a linear transformation $c \in \mathfrak{gl}(V)$ by the formula

$$c(v) = (X_+X_- + X_-X_+ + \frac{1}{2}H^2)v.$$

(a) Using the standard basis H, X_+, X_- of $\mathfrak{sl}(2)$, or otherwise, show that for any $a \in \mathfrak{sl}(2)$ we have $c(a(v)) = a(c(v))$.

(b) Determine c explicitly in the case of the tautological representation of $\mathfrak{sl}(2)$ in the vector space of column vectors $\mathbb{C}^2$.

(c) Determine c explicitly in the case of the adjoint representation of $\mathfrak{sl}(2)$.

3. In this question you can assume that the Killing form on the Lie algebra $\mathfrak{sl}(n)$ is a multiple of the trace form, i.e. there is a $\lambda \in \mathbb{C}$ such that for any $x, y \in \mathfrak{sl}(n)$ we have

$$K(x, y) = \lambda \operatorname{Tr}(xy).$$

Determine the value of λ.