Non-abelian descent and the arithmetic of Enriques surfaces

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Abstract. We give sufficient conditions for the composition of two torsors to be a torsor under the group which is an extension of the corresponding structure groups. This construction is applied to produce natural non-abelian torsors on the Enriques surfaces. We exhibit an Enriques surface over $\mathbf{Q}$ with a dense set of $\mathbf{Q}$-points, which is a counter-example to weak approximation accounted for by the descent obstruction defined by such a non-abelian torsor, but not by the Brauer–Manin obstruction.

Introduction

The Brauer–Manin obstruction to the Hasse principle and weak approximation provides a fruitful general approach to rational points on varieties over number fields. A fundamental problem here can be stated as follows: is it possible to describe in purely geometric terms the class of smooth projective varieties for which the Brauer–Manin obstruction is the only obstruction to the Hasse principle and weak approximation? In recent examples where the Brauer–Manin obstruction is not the only one ([12], [5], [1]) the key rôle is played by étale Galois coverings with a non-abelian Galois group. This has left open the question as to whether similar examples exist for varieties with an abelian geometric fundamental group. The case of principal homogeneous spaces of abelian varieties and that of rational surfaces (which are geometrically simply connected), where the Brauer–Manin obstruction is expected to be the only one, might seem to suggest that as long as the geometric fundamental group is abelian the Brauer–Manin obstruction should still be the only one.

The Manin obstruction was linked to the classical abelian descent by Colliot-Thélène and Sansuc [2]. In [6] the authors introduced the non-abelian descent as a new tool for studying rational points. The present paper enriches the non-abelian theory with a general method for constructing non-abelian torsors, and then applies it to an example which answers the above question in the negative.

The Enriques surfaces seem to lie close to the frontier separating the varieties whose arithmetic is controlled by the Brauer–Manin obstruction from those for which it is not the case. These surfaces are cohomologically indistinguishable from rational surfaces, but yet they possess a non-trivial geometric fundamental group $\mathbf{Z}/2$. We construct an Enriques surface over the field of rational numbers which is a counter-example to weak approximation that cannot be explained by the Brauer–Manin obstruction. More precisely, if a, b and c are integers satisfying some fairly mild conditions, then the quotient of the Kummer surface Y given by

$$y^2 = (x^2 - a)(x^2 - ab^2)(t^2 - a)(t^2 - ac^2)$$

by the involution which changes the signs of all the coordinates is an Enriques surface X with an adelic point which cannot be approximated by a rational point. However, this adelic point satisfies all the global reciprocity conditions provided by the elements of the Brauer group $\mathrm{Br}\, X$. (Note that in our example the Galois group acts trivially on $\mathrm{Pic}\, \overline{X}$.) For a numerical example we can choose $a = 5$, $b = 13$, $c = 2$. As an additional feature this particular Enriques surface has a Zariski dense set of rational points.

One way to look at this example is suggested by the philosophy of [12]: the natural map $\mathrm{Br}\, X \to \mathrm{Br}\, Y$ is not surjective; hence $\mathrm{Br}\, Y$ can impose more conditions on the adelic points in the closure of the set of rational points than $\mathrm{Br}\, X$. However, a more general interpretation is provided by non-abelian descent.

We show that a torsor over a K3 covering of an Enriques surface, which is stable under the action of the Enriques involution (for example, a universal torsor), can be considered as a non-abelian torsor over the Enriques surface. This is a particular case of a general situation when a composition of torsors is a torsor under the group which is an extension of relevant structure groups. This result reminiscent of Mumford's construction of theta-groups [8] is another main goal of this paper (see Theorem 1.2 and Proposition 1.4). It yields a large supply of non-abelian torsors; the non-abelian descent method can then be deployed with potential applications to weak approximation and the Hasse principle. The counter-example to weak approximation on the Enriques surface which we construct in this paper can be explained by the descent obstruction associated to this non-abelian torsor, as defined in [6]. Another application is a link between the approaches of [12] and [6]: we show that the 'iterated Manin obstruction' of [12] is in fact equivalent to the descent obstruction given by the composition of an abelian (e.g. universal) torsor with the corresponding étale Galois covering (see Proposition 1.6).

1. Composition of torsors

1.1. Preliminaries

Let k be a field of characteristic 0. Let $\bar{k}$ be an algebraic closure of k, $\Gamma = \mathrm{Gal}(\bar{k}/k)$. By a k-variety we understand in this paper a separated k-scheme of finite type. We write $\overline{X} = X \times_k \bar{k}$, and denote by $\bar{k}[X]^*$ the group of invertible regular functions on $\overline{X}$. A commutative algebraic k-group F *of multiplicative type* is an extension of a finite k-group by a k-torus. The *module of characters* $\widehat{F}$ of F is an abelian group of finite type acted on by Γ. If an algebraic k-group G acts on a k-variety Y preserving the fibres of a morphism $Y \to X$, then Y is a X-*torsor* under G if locally in the étale topology on X the variety Y with the action of G is isomorphic to the direct product $X \times_k G$. All cohomology groups in this paper are Galois or étale cohomology groups; we also consider the Galois cohomology set $H^1(k, G)$, where G is a not necessarily abelian algebraic k-group.

If Y is a geometrically integral variety such that $\bar{k}[Y]^* = \bar{k}^*$, and F is a k-group of multiplicative type, then there is the following exact sequence of Colliot-Thélène and Sansuc (see, e.g., [13], (2.22)):

$$0 \to H^1(k, F) \to H^1(Y, F) \xrightarrow{\chi} \mathrm{Hom}_\Gamma(\widehat{F}, \mathrm{Pic}\,\overline{Y}) \xrightarrow{\partial} H^2(k, F) \to H^2(Y, F). \quad (1)$$

If $Z \to Y$ is a torsor under F, then $\chi([Z]) \in \mathrm{Hom}_\Gamma(\widehat{F}, \mathrm{Pic}\,\overline{Y})$ is called the *type* of $Z \to Y$. When k is algebraically closed, then (1) shows that a torsor is determined by its type up to isomorphism. The variety Z is geometrically connected if and only if the kernel of $\chi([Z])$ has no torsion, for example when the type is injective. A Y-torsor under a group of multiplicative type is *universal* if its type is an isomorphism.

There is another useful exact sequence, also due to Colliot-Thélène and Sansuc ([2], (2.1.1)). Let T be a k-torus, and $Z \to Y$ a torsor under T, where both Y and Z are geometrically integral, and $\bar{k}[Y]^* = \bar{k}^*$. The following sequence of Γ-modules is then exact:

$$1 \to \bar{k}^* \to \bar{k}[Z]^* \to \widehat{T} \to \mathrm{Pic}\,\overline{Y} \to \mathrm{Pic}\,\overline{Z} \to 0. \quad (2)$$

Moreover, up to sign the map $\widehat{T} \to \mathrm{Pic}\,\overline{Y}$ coincides with the type of $Z \to Y$. It is clear from (2) that when the type is injective we have $\bar{k}[Z]^* = \bar{k}^*$.

In the case when $Z \to Y$ is a torsor under a finite k-group F, and the condition $\bar{k}[Z]^* = \bar{k}^*$ is satisfied, we still have an exact sequence ([13], (2.5))

$$0 \to \widehat{F} \to \mathrm{Pic}\,\overline{Y} \to \mathrm{Pic}\,\overline{Z}. \quad (3)$$

Here again, $\widehat{F} \to \mathrm{Pic}\,\overline{Y}$ is the type of the torsor $Z \to Y$ ([13], p. 25).

We write $\mathrm{Br}\,X$ for the cohomological Brauer–Grothendieck group $H^2(X, \mathbf{G}_m)$. Let $\mathrm{Br}_0 X = \mathrm{Im}\,[\mathrm{Br}\,k \to \mathrm{Br}\,X]$, $\mathrm{Br}_1 X = \mathrm{Ker}\,[\mathrm{Br}\,X \to \mathrm{Br}\,\overline{X}]$. The Hochschild–Serre spectral sequence (cf. [13], Cor. 2.3.9) defines a map $\mathrm{Br}_1 X \to H^1(k, \mathrm{Pic}\,\overline{X})$; if $\bar{k}[X]^* = \bar{k}^*$, then the kernel of this map is $\mathrm{Br}_0 X$. If $\lambda : M \to \mathrm{Pic}\,\overline{X}$ is a homomorphism of Γ-modules, then $\mathrm{Br}_\lambda X \subset \mathrm{Br}_1 X$ is the inverse image of $\lambda_* H^1(k, M) \subset H^1(k, \mathrm{Pic}\,\overline{X})$.

For a number field k we write Ω_k for the set of all places of k. Let $\mathbb{A}_k$ be the ring of adèles of k. For a subgroup $B \subset \mathrm{Br}\,X$ define

$$X(\mathbb{A}_k)^B = \big\{ \{P_v\} \in X(\mathbb{A}_k) \mid \sum_{v \in \Omega_k} \mathrm{inv}_v(\alpha(P_v)) = 0, \ \ \forall \alpha \in B \big\},$$

where $X(\mathbb{A}_k)$ is the set of adelic points of X, and $\mathrm{inv}_v : \mathrm{Br}\, k_v \to \mathbf{Q}/\mathbf{Z}$ is the local invariant of the local class field theory. By global reciprocity we have $X(k) \subset X(\mathbb{A}_k)^{\mathrm{Br}}$. When X is proper $X(\mathbb{A}_k)^{\mathrm{Br}}$ contains the closure $\overline{X(k)}$ of $X(k)$ in $X(\mathbb{A}_k) = \prod_{v\in\Omega_k} X(k_v)$ in the product topology.

Finally, for a torsor $f : Z \to X$ under a k-group G we write

$$X(\mathbb{A}_k)^f = \left\{ \{P_v\} \in X(\mathbb{A}_k) \;\middle|\; \{[Z_{P_v}]\} \in \mathrm{Im}[H^1(k,G) \to \prod_{v\in\Omega_k} H^1(k_v,G)] \right\}.$$

We have $X(k) \subset X(\mathbb{A}_k)^f$; moreover, $\overline{X(k)} \subset X(\mathbb{A}_k)^f$ when X is proper and G is linear (see [13], Prop. 5.3.3).

1.2. A general result

We shall need the following auxiliary statement.

Lemma 1.1 *Let $Y \to X$ be a torsor under an algebraic k-group G. Assume that the image of any $\bar{k}$-morphism $\overline{Y} \to \overline{G}$ is a $\bar{k}$-point. Then $G(\bar{k}) = \mathrm{Aut}\,(\overline{Y}/\overline{X})$.*

Proof. Let $\psi \in \mathrm{Aut}\,(\overline{Y}/\overline{X})$. The canonical isomorphism $\overline{Y} \times_{\overline{X}} \overline{Y} = \overline{Y} \times_{\bar{k}} \overline{G}$ identifies the graph of ψ with the graph of a morphism $g : \overline{Y} \to \overline{G}$. Now by assumption we have $g(y) = g_0$ for some $g_0 \in G(\bar{k})$ and any $y \in Y(\bar{k})$. Hence $\psi(y) = g_0 y$. QED

Colliot-Thélène pointed out to us that the converse is false, e.g. for $X = \mathrm{Spec}\, k$, $Y = \mathrm{Spec}\,(k \oplus k)$ with the action of $G = \mathbf{Z}/2$ by permutations.

The main result of this section is the following

Theorem 1.2 *Let F and H be algebraic k-groups, $p : Z \to Y$ a torsor under F, and $Y \to X$ a torsor under H, where X is a smooth and geometrically integral k-variety. Assume the following conditions:*

1. *For each $h \in H(\bar{k})$ there exists an isomorphism of $\bar{k}$-varieties $\varphi_h : \overline{Z} \to \overline{Z}$ such that the following diagram is commutative:*

$$\begin{array}{ccc} \overline{Z} & \xrightarrow{\varphi_h} & \overline{Z} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ \overline{Y} & \xrightarrow{h} & \overline{Y} \end{array}$$

2. *The image of any $\bar{k}$-morphism $\overline{Z} \to \overline{F}$ is a $\bar{k}$-point.*

Then there is an exact sequence of algebraic k-groups

$$1 \to F \to G \to H \to 1 \qquad (4)$$

such that the action of F on Z extends to an action of G which induces the action of H on the quotient $Y = Z/F$. This action makes Z into an X-torsor under G.

Therefore, the theorem gives a natural sufficient condition for a composition of two torsors to be a torsor. In the proof G is constructed in a certain canonical way, namely, $G(\bar{k})$ is the group of $\bar{k}$-automorphisms of $\overline{Z}$ which are liftings of the automorphisms of $\overline{Y}$ defined by the elements of $H(\bar{k})$.

Proof of the theorem. Define $\mathcal{G}$ as the subset of $\mathrm{Aut}_{\bar{k}}(\overline{Z})$ consisting of the automorphisms ϕ such that there exists $h \in H(\bar{k})$ making the diagram

$$\begin{tikzcd} \overline{Z} \arrow[r, "\phi"] \arrow[d, "p"'] & \overline{Z} \arrow[d, "p"] \\ \overline{Y} \arrow[r, "h"] & \overline{Y} \end{tikzcd}$$

commutative. To any $\phi \in \mathcal{G}$ there corresponds exactly one $h \in H(\bar{k})$ because the action of H on Y is faithful. Since $(h, y) \mapsto hy$ is an action of H on Y, we see that $\mathcal{G}$ is a subgroup of $\mathrm{Aut}_{\bar{k}}(\overline{Z})$. For the same reason the natural map $\pi : \mathcal{G} \to H(\bar{k})$ is a homomorphism. Obviously $F(\bar{k})$ is a subgroup of $\mathcal{G}$ contained in the kernel of π. By Lemma 1.1 condition 2 of the theorem implies that $\mathrm{Aut}\,(\overline{Z}/\overline{Y}) = F(\bar{k})$, hence we obtain an exact sequence of groups

$$1 \to F(\bar{k}) \to \mathcal{G} \xrightarrow{\pi} H(\bar{k}).$$

The $\bar{k}$-varieties $\overline{Z}$ and $\overline{X}$ come from varieties defined over k, therefore there is a natural action of Γ on the group $\mathrm{Aut}\,(\overline{Z}/\overline{X})$; this action is defined by the formula (cf. [10], III.1.1):

$$({}^{\gamma}\varphi)(z) = {}^{\gamma}(\varphi({}^{\gamma^{-1}}z)), \quad \gamma \in \Gamma, \quad z \in Z(\bar{k}), \quad \varphi \in \mathrm{Aut}\,(\overline{Z}/\overline{X}). \tag{5}$$

One checks immediately that the commutativity of the diagram above implies the commutativity of the same diagram with ϕ and h replaced by ${}^{\gamma}\phi$ and ${}^{\gamma}h$, respectively. This shows that the subgroup $\mathcal{G} \subset \mathrm{Aut}_{\bar{k}}(\overline{Z})$ is stable under the action of the Galois group Γ.

Lemma 1.3 *Let $z_0 \in Z(\bar{k})$. The map $\theta : \mathcal{G} \to p^{-1}(H(\bar{k}).p(z_0))$ defined by $g \mapsto gz_0$ is a bijection.*

Proof. θ is injective. Suppose that $g_1 z_0 = g_2 z_0$. This implies in particular that g_1 and g_2 are mapped to the same $h \in H(\bar{k})$, that is, $g_1 g_2^{-1}$ is in $\mathrm{Aut}\,(\overline{Z}/\overline{Y})$. Since $\mathrm{Aut}\,(\overline{Z}/\overline{Y}) = F(\bar{k})$ we have $g_1 g_2^{-1} \in F(\bar{k})$. But $g_1 g_2^{-1}$ fixes z_0, thus $g_1 = g_2$.

θ is surjective. Let $z_1 \in p^{-1}(H(\bar{k}).p(z_0))$. Then there exists $h \in H(\bar{k})$ such that $hp(z_0) = p(z_1)$. Let φ_h be a $\bar{k}$-automorphism of $\overline{Z}$ such that $p \circ \varphi_h = h \circ p$. Then $p(\varphi_h(z_0)) = p(z_1)$. It remains to modify φ_h by an element of $F(\bar{k})$ to obtain $\phi \in \mathcal{G}$ such that $\phi(z_0) = z_1$. QED

End of the proof of the theorem. We have an obvious commutative diagram

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\theta} & p^{-1}(H(\bar{k}).p(z_0)) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle p} \\ H(\bar{k}) & \longrightarrow & H(\bar{k}).p(z_0) \end{array}$$

The map $H(\bar{k}) \to H(\bar{k}).p(z_0)$ is bijective because the action of H on Y is free. This together with the bijectivity of θ shows that $\pi : \mathcal{G} \to H(\bar{k})$ is surjective. We now have a Galois equivariant extension of groups

$$1 \to F(\bar{k}) \to \mathcal{G} \to H(\bar{k}) \to 1.$$

Let $\overline{G}$ be the $\bar{k}$-variety $p^{-1}(H(\bar{k}).p(z_0))$. The bijection θ makes $\overline{G}$ into an algebraic $\bar{k}$-group, an extension of $\overline{H}$ by $\overline{F}$. Since the group variety $\overline{G}$ is quasi-projective and the action of Γ on it is continuous, we can define G as the quotient of $\overline{G}$ by this action. It is clear that G is an extension of H by F, and that G acts on Z with required properties. QED

Remark. We see from Lemma 1.1 that when the connected component of G is a torus, and $\bar{k}[Z]^* = \bar{k}^*$, the group $G(\bar{k})$ coincides with $\mathrm{Aut}\,(\overline{Z}/\overline{X})$ equipped with a natural action of Γ.

The following proposition gives sufficient conditions for Theorem 1.2 which are easy to verify.

Proposition 1.4 *Let $p : Z \to Y$ be a torsor under a k-group F of multiplicative type, and $Y \to X$ be a torsor under an algebraic k-group H. Assume that X is a smooth and geometrically integral k-variety, and that Y is such that $\bar{k}[Y]^* = \bar{k}^*$ (e.g. proper and geometrically connected). Assume also that the type $\lambda \in \mathrm{Hom}_\Gamma(\widehat{F}, \mathrm{Pic}\,\overline{Y})$ of the torsor $p : Z \to Y$ is injective with $H(\bar{k})$-invariant image (e.g. the torsor is universal). Then condition 1 of the theorem is satisfied. Condition 2 is satisfied as long as F is finite, or F is a torus, or Y is proper.*

Proof. The action of H on Y defines a natural Γ-equivariant action of $H(\bar{k})$ on $\mathrm{Pic}\,\overline{Y}$ and the assumption we made about the type λ implies that this action gives rise to a natural Γ-equivariant action $\hat{\tau}$ of $H(\bar{k})$ on $\widehat{F}$; we have $\lambda \circ \hat{\tau}_h = h^* \circ \lambda$ for each $h \in H(\bar{k})$. Since Y satisfies the condition $\bar{k}[Y]^* = \bar{k}^*$, a $\overline{Y}$-torsor under $\overline{F}$ is uniquely determined up to isomorphism by its type. Now the formula above shows that the $\overline{Y}$-torsor under $\overline{F}$ obtained from $\overline{Z} \to \overline{Y}$ by the base change $h : \overline{Y} \to \overline{Y}$ has the same type as the $\overline{Y}$-torsor under $\overline{F}$ obtained from $\overline{Z} \to \overline{Y}$ by the transformation of structure group $\tau_h : \overline{F} \to \overline{F}$. Hence these torsors are isomorphic, by (1). Therefore, any $h \in H(\bar{k})$ lifts to an automorphism φ_h of the $\bar{k}$-variety $\overline{Z}$, which means that condition 1 of the theorem is satisfied.

Since λ is injective, the variety Z is geometrically connected. Thus for the finite F condition 2 of the theorem is obvious.

If F is a torus, then the injectivity of the map λ implies that $\bar{k}[Z]^* = \bar{k}^*$, whence $\mathrm{Mor}_{\bar{k}}(\overline{Z}, \overline{F}) = F(\bar{k})$.

Finally we consider the case when Y is proper. The k-group F is an extension of a finite commutative k-group F_1 by a torus T. Let $Y_1 = Z/T$; this proper variety is a Y-torsor under F_1. The functoriality of (1) with respect to the change of the structure group $F \to F_1$ implies that the type of $Y_1 \to Y$ is the composition

$$\widehat{F}_1 \hookrightarrow \widehat{F} \hookrightarrow \mathrm{Pic}\,\overline{Y},$$

hence is injective. Therefore Y_1 is geometrically connected, hence $\bar{k}[Y_1]^* = \bar{k}^*$. Next, Z is a Y_1-torsor under T. The following diagram commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widehat{F}_1 & \longrightarrow & \widehat{F} & \longrightarrow & \widehat{T} & \longrightarrow & 0 \\
 & & || & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \widehat{F}_1 & \longrightarrow & \mathrm{Pic}\,\overline{Y} & \longrightarrow & \mathrm{Pic}\,\overline{Y}_1 & &
\end{array}$$

The top line here is obvious, and the bottom one is the sequence (3) defined by the torsor $Y_1 \to Y$. The middle vertical arrow is the type of $Z \to Y$, and right hand one is the type of $Y_1 \to Y$. The left hand square commutes by the functoriality of type with respect to the structure group change $F \to F_1$. To prove that the right hand square commutes we note that the push-forward of the torsor $Z \to Y_1$ with respect to the morphism of the structure group $T \to F$ gives the same F-torsor as the pull-back of the F-torsor $Z \to Y$ to Y_1. Indeed, the push-forward is the quotient $(Z \times_k F)/T$, where T acts by sending (z, f) to $(t^{-1}z, tf)$. Here the structure of an F-torsor is obtained from the action of F on the second factor. The canonical isomorphism $Z \times_k F = Z \times_Y Z$ translates the action of T into $(z_1, z_2) \mapsto (t^{-1}z_1, z_2)$. The quotient by this action is $Y_1 \times_Y Z$, the pull-back of $Z \to Y$ to Y_1. We now see that the two different ways to build a map $\widehat{F} \to \mathrm{Pic}\,\overline{Y}_1$ in the diagram coincide since both are equal to the type of the F-torsor $Y_1 \times_Y Z \to Y_1$. This establishes the commutativity.

An easy diagram chase now shows that the map $\widehat{T} \to \mathrm{Pic}\,\overline{Y}_1$ is injective. By the remarks after (2) we have $\bar{k}[Z]^* = \bar{k}^*$. Since $\overline{F}$ is of multiplicative type, this implies that $\mathrm{Mor}_{\bar{k}}(\overline{Z}, \overline{F}) = F(\bar{k})$. QED

1.3. Examples

Example 1. Mumford's theta-groups. Let L be a line bundle on an abelian variety A. The complement to the zero section of L is an A-torsor under the multiplicative group $\mathbf{G}_m$. Let $K(L) \subset A$ be the closed subscheme whose closed points are the elements $a \in A(\bar{k})$ such that L is isomorphic to a^*L, the translation of L by a. Note that $K(L)$ is finite if and only if L is ample ([8], II.6, Prop. 1). The assumptions of Theorem 1.2 are satisfied because of Proposition 1.4. The resulting extension of $K(L)$ by $\mathbf{G}_m$ is a theta-group; these groups have numerous beautiful applications, see [8], VI.23.

Example 2. Let A be an abelian variety with an action of a finite group scheme H. Let us assume that the group scheme A^H (the points of A fixed by H) is finite. Let Y be a principal homogeneous space of A such that the class $[Y] \in H^1(k, A)$ comes from $H^1(k, A^H)$. Then H naturally acts on Y. Let D be a projective variety with a free action of H. The simultaneous action of H on $Y \times_k D$ is free. Let $X = (Y \times_k D)/H$.

Let $\alpha : A \to B = A/A^H$ be the natural isogeny. Choose a positive integer m such that $A^H \subset A[m]$. The multiplication by m map factors through α, so that we can write $m = \beta \circ \alpha$, where $\beta : B \to A$ is an isogeny. Suppose that Z is a principal homogeneous space of B such that $[Y] = \beta_*[Z]$. Then there is a natural push-forward map $Z \to Y$ (quotient by $\mathrm{Ker}\,(\beta) = A[m]/A^H$). This map makes Z into a Y-torsor under the group scheme $A[m]/A^H$.

Let A^t be the dual abelian variety of A. The dual map to the injection $A^H \to A[m]$ is the surjection $A^t[m] \to \widehat{A^H}$. Let F be its kernel; this is the group dual to $A[m]/A^H$. The type of the torsor $Z \times_k D \to Y \times_k D$ under $A[m]/A^H$ (which acts trivially on D) is the composed map

$$F \to A^t[m] \to A^t(\bar{k}) = \mathrm{Pic}_0\overline{Y} \to \mathrm{Pic}\,\overline{Y} \to \mathrm{Pic}\,(\overline{Y} \times_{\bar{k}} \overline{D}).$$

From the theorem we obtain that $Z \times_k D$ is an X-torsor under a finite k-group G which is an extension of H by F.

The simplest case is when $H = \mathbf{Z}/2$, and the non-trivial element of H acts on A as multiplication by -1. Then $A^H = A[2]$, Y is any principal homogeneous space of A such that $2[Y] = 0$, m is any positive even number, $B = A$, α is the multiplication by 2 map. If A is an elliptic curve, and D is a curve of genus 1 on which the non-trivial element of H acts as a translation, then X is a bielliptic surface. Its curious arithmetic properties were studied in [3] and [12]. An important rôle in [12] was played by a torsor $Z \times_k D \to X$ with $m = 8$ (then G is non-abelian). The group $H = \mu_3$ leads to bielliptic surfaces of a different type; their arithmetic was studied in [1].

Example 3. Let X be an Enriques surface over k, and $Y \to X$ be a K3-covering of X. Let Z be a universal Y-torsor. It is a torsor under the Néron–Severi torus T of Y. The theorem then says that Z is an X-torsor under a k-group G which is an extension of $\mathbf{Z}/2$ by T.

The group G is commutative if and only if the natural map $\mathrm{Pic}\,\overline{X} \to \mathrm{Pic}\,\overline{Y}$ is surjective. (This can be checked over $\bar{k}$.) Indeed, the exact sequence in the proof of

Lemma 2.4 below shows that $\mathbf{Z}/2$ acts trivially on $\operatorname{Pic}\overline{Y}$ if and only if $\operatorname{Pic}\overline{X} \to \operatorname{Pic}\overline{Y}$ is surjective. In this case $\mathbf{Z}/2$ also acts trivially on $\overline{T}$. Since $H^2(\mathbf{Z}/2, \bar{k}^*) = 0$ for the trivial $\mathbf{Z}/2$-module structure on $\bar{k}^*$, the extension is a semi-direct product. Because of the trivial action we have $\overline{G} = \overline{T} \times \mathbf{Z}/2$. Conversely, if G is commutative, then $\mathbf{Z}/2$ acts trivially on $\operatorname{Pic}\overline{Y}$, hence $\operatorname{Pic}\overline{X} \to \operatorname{Pic}\overline{Y}$ is surjective.

1.4. Non-abelian torsor obstruction versus Manin obstruction on abelian torsors

In this subsection we clarify the relation between the (non-abelian) torsor obstruction [6] and the 'iterated Manin obstruction' [12]. All varieties are assumed to be smooth and quasi-projective. For more details on twisted forms of groups and torsors see [13], Ch. 2.

Lemma 1.5 *Let G' be a k-form of an algebraic k-group G. Let $Z \to X$ be a torsor under G, and $Z' \to X$ be a torsor under G'. Suppose that the torsors $\overline{Z} \to \overline{X}$ and $\overline{Z}' \to \overline{X}$ under $\overline{G} = \overline{G}'$ are isomorphic. Assume that every morphism from $\overline{Z}$ to $\overline{G}$ maps $\overline{Z}$ to a point. Then there exists a continuous 1-cocycle ρ of Γ with coefficients in $G(\bar{k})$ such that $G' = G^\rho$ is the inner form of G defined by ρ, and $Z' = Z^\rho$ is the twisted form of Z defined by ρ with respect to the natural action of G on Z.*

Proof. The k-form G' of G defines a 'twisted' action of Γ on $\overline{G}$, denoted by ${}^{\gamma\star}g$ as opposed to the standard action ${}^{\gamma}g$, where $\gamma \in \Gamma$, $g \in G(\bar{k})$. Choose an isomorphism of $\overline{X}$-torsors under $\overline{G}$, $\overline{Z} \simeq \overline{Z}'$. Then Z' defines a 'twisted' action of Γ on $\overline{Z}$, denoted by ${}^{\gamma\star}z$ as opposed to the standard action ${}^{\gamma}z$, where $\gamma \in \Gamma$, $z \in Z(\bar{k})$. The points ${}^{\gamma\star}z$ and ${}^{\gamma}z$ belong to the same fibre of $\overline{Z} \to \overline{X}$. Hence ${}^{\gamma\star}z = g(z,\gamma) \cdot {}^{\gamma}z$, where, for a fixed γ, $g(z,\gamma)$ is a morphism from $\overline{Z}$ to $\overline{G}$. By our assumption $g(z,\gamma)$ does not depend on z. We write $g(z,\gamma) = g(\gamma)$. It is clear that this is a locally constant, hence continuous function $\Gamma \to G(\bar{k})$. (Recall that Γ has natural profinite topology, and $G(\bar{k})$ has discrete topology.) Let $g \in G(\bar{k})$, $z \in Z(\bar{k})$. Then

$${}^{\gamma\star}(gz) = g(\gamma) \cdot {}^{\gamma}(gz) = g(\gamma) \cdot {}^{\gamma}g \cdot {}^{\gamma}z.$$

On the other hand,

$${}^{\gamma\star}(gz) = {}^{\gamma\star}g \cdot {}^{\gamma\star}(z) = {}^{\gamma\star}g \cdot g(\gamma) \cdot {}^{\gamma}z.$$

Since $\overline{G}$ acts freely on $\overline{Z}$, we have

$${}^{\gamma\star}g = g(\gamma) \cdot {}^{\gamma}g \cdot g(\gamma)^{-1}. \tag{6}$$

We have ${}^{(\gamma_1\gamma_2)\star}z = g(\gamma_1\gamma_2) \cdot {}^{\gamma_1\gamma_2}z$. But this also equals

$${}^{\gamma_1\star}({}^{\gamma_2\star}z) = {}^{\gamma_1\star}(g(\gamma_2) \cdot {}^{\gamma_2}z) = {}^{\gamma_1\star}(g(\gamma_2)) \cdot g(\gamma_1) \cdot {}^{\gamma_1\gamma_2}z.$$

Substituting (6) we deduce that $g(\gamma_1\gamma_2) = g(\gamma_1) \cdot {}^{\gamma_1}g(\gamma_2)$ which says that $g(\gamma)$ is a 1-cocycle of Γ with coefficients in $G(\bar{k})$. Formula (6) now shows that G' is indeed the inner form of G defined by this cocycle. Furthermore, Z' is the twist of Z defined by the same cocycle. QED

In the notation of Proposition 1.4 let σ be a continuous 1-cocycle of Γ with coefficients in $H(\bar{k})$. Various objects acted on by H can be twisted by σ. We thus obtain the twisted k-variety Y^σ and the twisted k-group of multiplicative type F^σ. The natural action τ of H on F, constructed in the beginning of the proof of Proposition 1.4, comes from the natural Γ-equivariant action of $H(\bar{k})$ on $\mathrm{Pic}\,\overline{Y}$. By construction this action of $H(\bar{k})$ preserves the injection of Γ-modules $\lambda : \widehat{F} \hookrightarrow \mathrm{Pic}\,\overline{Y}$. Thus after twisting we obtain a natural injection of Γ-modules $\lambda^\sigma : \widehat{F}^\sigma \hookrightarrow \mathrm{Pic}\,\overline{Y}^\sigma$.

Let H^σ be the inner form of H defined by σ. That is, H^σ is the algebraic k-group obtained from H by twisting it by σ with respect to the action of H on itself by conjugations. The group H^σ acts on Y^σ so that the natural morphism $r^\sigma : Y^\sigma \to X$ is a torsor under H^σ. We also have a natural Γ-equivariant action of $H^\sigma(\bar{k})$ on $\mathrm{Pic}\,\overline{Y}^\sigma$, and an action of H^σ on F^σ.

Proposition 1.6 *Let k be a number field. Let F be a k-group of multiplicative type, and H be a finite k-group. Let $r : Y \to X$ be a torsor under H. Let $p : Z \to Y$ be a torsor under F whose type is injective with $H(\bar{k})$-invariant image, satisfying $\bar{k}[Z]^* = \bar{k}^*$. Then the conditions of Proposition* 1.4 *are satisfied. Let $f : Z \to X$ be the torsor under G obtained by composing torsors $p : Z \to Y$ and $r : Y \to X$ as in Theorem* 1.2. *Then*

$$X(\mathbb{A}_k)^f = \bigcup_{[\sigma]\in H^1(k,H)} r^\sigma(Y^\sigma(\mathbb{A}_k)^{\mathrm{Br}_{\lambda^\sigma}}). \tag{7}$$

In particular, if $Z \to Y$ is a universal torsor, then

$$X(\mathbb{A}_k)^f = \bigcup_{[\sigma]\in H^1(k,H)} r^\sigma(Y^\sigma(\mathbb{A}_k)^{\mathrm{Br}_1}).$$

It can be shown that the injectivity of the type of $Z \to Y$ is a consequence of other conditions.

Proof of the proposition. The conditions of Proposition 1.4 are satisfied since $\bar{k}[Z]^* = \bar{k}^*$ implies $\bar{k}[Y]^* = \bar{k}^*$. The left hand side of (7) is

$$X(\mathbb{A}_k)^f = \bigcup_{[\xi]\in H^1(k,G)} f^\xi(Z^\xi(\mathbb{A}_k)),$$

(cf. [13], Def. 5.3.1). Here $f^\xi : Z^\xi \to X$ is the twisted torsor of $f : Z \to X$ by a continuous 1-cocycle ξ of Γ with coefficients in $G(\bar{k})$. Let σ be the image of ξ with respect to the surjective morphism of algebraic k-groups $G \to H$. There is a natural surjective map (quotient by F^σ) of $Z^\xi \to Y^\sigma$, where Y^σ, defined as the twist of Y, is an X-torsor under H^σ. The map $Z^\xi \to Y^\sigma$ makes Z^ξ into a Y^σ-torsor under F^σ of type λ^σ.

Let us turn to the right hand side. By the main result of the descent theory of Colliot-Thélène and Sansuc (see [13], Thm. 6.1.2) we have

$$Y^\sigma(\mathbb{A}_k)^{\mathrm{Br}_{\lambda^\sigma}} = \cup\, p'(Z'(\mathbb{A}_k)),$$

where $p' : Z' \to Y^\sigma$ ranges over Y^σ-torsors under F^σ of type λ^σ. The conditions of Proposition 1.4 are satisfied and we can compose the torsors $r^\sigma \circ p' : Z' \to Y^\sigma \to X$. Thus Z' is an X-torsor under a certain k-group G' which is an extension of H^σ by F^σ.

We observe that $p : \overline{Z} \to \overline{Y}$ and $p' : \overline{Z}' \to \overline{Y}^\sigma = \overline{Y}$ have the same type as $\overline{Y}$-torsors under $\overline{F}$. Therefore, these torsors are isomorphic. The group $G(\bar{k})$ was constructed in the proof of Theorem 1.2 as the group of the automorphisms of $\overline{Z}$ over $\overline{Y}$ which are liftings of the elements of $H(\bar{k})$. The structure of an algebraic variety on $G(\bar{k})$ was defined via its identification with $p^{-1}(H(\bar{k}).y_0)$, for some $y_0 \in Y(\bar{k})$. We conclude that the $\bar{k}$-groups $\overline{G}$ and $\overline{G}'$ are isomorphic. Thus G' is a k-form of the algebraic k-group G.

The assumption $\bar{k}[Z]^* = \bar{k}^*$ implies that Z is geometrically connected. The connected component of G is a torus, hence the same assumption shows that the image of any morphism $\overline{Z} \to \overline{G}$ is a point. By Lemma 1.5 for some continuous 1-cocycle $\xi : \Gamma \to G(\bar{k})$ we have $Z' = Z^\xi$, $G' = G^\xi$. Comparing the formula ${}^{\gamma\star}z = \xi(\gamma) \cdot {}^{\gamma}z$ from the proof of Lemma 1.5 with the analogous formula for Y^σ we see that map $G \to H$ sends ξ to σ.

We have proved that every Y^σ-torsor under F^σ of type λ^σ is isomorphic to Z^ξ, for some lifting $[\xi] \in H^1(k, G)$ of $[\sigma] \in H^1(k, H)$. This completes the proof of (7). QED

2. Enriques surface of Kummer type

2.1. Constructions

Let E_1, E_2 be elliptic curves over k which are not isogenous over $\bar{k}$, and such that their points of order 2 are defined over k. For $i = 1, 2$ let D_i be a principal homogeneous space of E_i whose class in $H^1(k, E_1)$ has order 2. The antipodal involution $P \mapsto -P$ on E_i defines an involution on D_1 and on D_2. We shall denote all these involutions by ι.

Let Y be the Kummer surface built from $D_1 \times D_2$. This is the minimal desingularization of the quotient of $D_1 \times D_2$ by the simultaneous antipodal involution.

Lemma 2.1 *Let $P \in E_1[2]$, $Q \in E_2[2]$. The involution of $D_1 \times D_2$ given by $(x, y) \mapsto (x + P, \iota(y) + Q)$ descends to an involution $\sigma : Y \to Y$ without fixed points.*

Proof. Note that $(x, y) \mapsto (x + P, \iota(y) + Q)$ commutes with the involution $(x, y) \mapsto (\iota(x), \iota(y))$. This rule defines an involution on the singular surface $(D_1 \times D_2)/\iota$, hence also on its minimal desingularization Y. QED

Recall that the quotient of a K3 surface by any fixed point free involution is an Enriques surface. The lemma thus allows us to define an Enriques surface $X = Y/\sigma$. Let $f : Y \to X$ be the corresponding unramified double covering.

For our purposes we shall consider the following simplest case of the above construction. Let $a \in k^* \setminus k^{*2}$, and let b, c, d_1, d_2 be in k^* such that $b \neq \pm 1$, $c \neq \pm 1$. Let the curves D_1 and D_2 be given by their respective (affine) equations:

$$y_1^2 = d_1(x^2 - a)(x^2 - ab^2), \qquad y_2^2 = d_2(t^2 - a)(t^2 - ac^2).$$

The antipodal involution changes the signs of y_1 (resp. of y_2). Hence the Kummer surface Y is the minimal, smooth and projective model of the affine surface

$$y^2 = d(x^2 - a)(x^2 - ab^2)(t^2 - a)(t^2 - ac^2), \tag{8}$$

where $y = y_1 y_2$, $d = d_1 d_2$.

When k is a number field it is not hard to give a sufficient condition that guarantees that E_1 and E_2 are not isogenous over $\bar{k}$.

Lemma 2.2 *Let k be a number field with the ring of integers $\mathcal{O}_k$. For any prime $\wp \in \mathrm{Spec}\,(\mathcal{O}_k)$ let $v_\wp : k^* \to \mathbf{Z}$ be the associated valuation. Assume that there exists a prime $\wp \in \mathrm{Spec}\,(\mathcal{O}_k)$ not dividing 2 such that*

$$v_\wp(b) > 0, \quad v_\wp(c) = v_\wp(c - 1) = v_\wp(c + 1) = 0.$$

Then E_1 and E_2 are not isogenous over $\bar{k}$.

Proof. The change of variables $u = a^{-1} y_1 x$, $t = a^{-1} x^2$ gives a degree 2 morphism $D_1 \to E_1'$, where E_1' is the elliptic curve with equation $u^2 = ad_1 t(t - 1)(t - b^2)$. This implies that E_1 and E_1' are isogenous. The curve E_1' is a quadratic twist of the elliptic curve

$$E_1'' : u^2 = t(t - 1)(t - b^2).$$

In particular, $\overline{E}_1$ and $\overline{E}_1''$ are isogenous. The same argument shows that E_2 is $\bar{k}$-isogenous to the elliptic curve E_2'' given by $u^2 = t(t - 1)(t - c^2)$. The j-invariant of E_1'' equals

$$j_1 = 2^8 \frac{(b^4 - b^2 + 1)^3}{b^4 (b^2 - 1)^2}$$

(see, e.g. [5], p. 317), and the j-invariant j_2 of E_2'' is given by a similar formula. Our assumptions imply that $v_\wp(j_2) \geq 0$ and $v_\wp(j_1) < 0$. In particular, j_1 is not integral over the ring $\mathbf{Z}[j_2]$. By Theorem 2.6.3 of [11] the curves E_1'' and E_2'' are not isogenous over $\bar{k}$. Hence the same is true for E_1 and E_2. QED

Example. If the ranks of E_1 and E_2 are positive, then k-points are Zariski dense on $E_1 \times E_2$. Take $D_1 = E_1$, $D_2 = E_2$. Then k-points are dense on Y and hence on X. This is a simple way to construct Enriques surfaces over a number field k with a Zariski dense set of k-points (cf. [14]). For example, let $k = \mathbf{Q}$ and $a = 5$, $b = 13$, $c = 2$, $d_1 = d_2 = 1$. Then as in the previous proof $E_1 \simeq D_1$ is isogenous to the curve $y^2 = x(x-5)(x-845)$ which has a point $(4, 58)$ of infinite order. Similarly, $E_2 \simeq D_2$ is isogenous to the curve $y^2 = x(x-5)(x-20)$ with a point $(4, 8)$ of infinite order. Applying Lemma 2.2 with $\wp = 13$ we see that E_1 and E_2 are not isogenous over $\overline{\mathbf{Q}}$.

To get an explicit expression of the Enriques involution σ let P (resp. Q) be the point of order 2 in E_1 (resp. E_2) given by the difference of two points $(\sqrt{a}, 0) - (-\sqrt{a}, 0)$ on D_1 (resp. D_2). Then σ as defined in Lemma 2.1 is

$$\sigma(x, t, y) = (-x, -t, -y). \tag{9}$$

Indeed, the translation $t_P : D_1 \to D_1$ by P commutes with the antipodal involution, hence it descends to the quotient by the antipodal involution, that is, to $\mathbf{P}^1_k$ with coordinate x. Thus the x-coordinate of $t_P(x, y_1)$ is $\phi(x)$, where $\phi \in PGL(2, k)$. We note that ϕ swaps $\sqrt{a}$ and $-\sqrt{a}$, and also $b\sqrt{a}$ and $-b\sqrt{a}$. Since the elements of $PGL(2, k)$ are uniquely determined by the action on any four pairwise distinct points of $\mathbf{P}^1_k$, we conclude that $\phi(x) = -x$. Therefore, $t_P(x, y_1)$ is either $(-x, y_1)$ or $(-x, -y_1)$. Since t_P has no fixed points we must have $t_P(x, y_1) = (-x, -y_1)$. Similarly, $t_Q \circ \iota(t, y_2) = (-t, y_2)$. This implies (9). One can also directly check that (9) defines a fixed point free involution on Y.

We enumerate the points of D_1 with coordinates $(\sqrt{a}, 0)$, $(-\sqrt{a}, 0)$, $(b\sqrt{a}, 0)$, $(-b\sqrt{a}, 0)$ by $i = 0, 1, 2, 3$. Similarly, the points of D_2 with coordinates $(\sqrt{a}, 0)$, $(-\sqrt{a}, 0)$, $(c\sqrt{a}, 0)$, $(-c\sqrt{a}, 0)$ are numbered by $j = 0, 1, 2, 3$. Let l_{ij} be the smooth, proper, rational curve on $\overline{Y}$ that is the blowing-up of the image of (i, j) on $(\overline{D}_1 \times \overline{D}_2)/\iota$. Let l_i (resp. s_j) be the proper transform of the rational curve $(i \times \overline{D}_2)/\iota$ (resp. $(\overline{D}_1 \times j)/\iota$) on $\overline{Y}$. The non-zero intersection indices of these 24 projective lines on $\overline{Y}$ can be listed as follows:

$$(l_i . l_{ij}) = 1, \;\; (l_{ij} . s_j) = 1.$$

Consider the morphisms $\pi_i : Y \to D_i/\iota = \mathbf{P}^1_k$, $i = 1, 2$. Explicitly, π_1 (resp. π_2) is given by the projection to the coordinate x (resp. t). The smooth fibres of π_1 (resp. of π_2) are curves of genus 1, and the singular fibres correspond to the points with $y_1 = 0$ (resp. $y_2 = 0$). Let f_1 (resp. f_2) be the (smooth) fibre of π_1 (resp. of

π_2) at $x=\infty$ (resp. $t=\infty$). The structure of singular fibres is such that for any $i,j\in\{1,2,3,4\}$ we have the following relations in $\mathrm{Pic}\,\overline{Y}$:

$$[f_1]=2[l_i]+\sum_k[l_{ik}],\quad [f_2]=2[s_j]+\sum_k[l_{kj}].$$

We note an important relation which is straightforward to verify:

$$\mathrm{div}\,(y)=\sum l_i+\sum s_j+\sum l_{ij}-2f_1-2f_2. \tag{10}$$

Let $U\subset D_1\times D_2$ be the complement to the 16 points with $y_1=y_2=0$. Then $V=U/\iota$ is the complement to the 16 lines l_{ij} in Y. We have an unramified double covering $U\to V$. We shall also need smaller open sets $U'=(D_1\setminus\{y_1=0\})\times(D_2\setminus\{y_2=0\})$, and its quotient $V'=U'/\iota$, which is the complement to the 24 lines in Y.

Let $L=k(\sqrt{a})$. We make an important observation that *the* 24 *lines of* $\overline{Y}$ *are defined over the quadratic extension* L/k, *and that the action of the Enriques involution* σ *on the* 24 *lines coincides with the action of the Galois group* $\mathrm{Gal}\,(L/k)$. This fact will simplify subsequent computations of various cohomology groups.

The following proposition explains the rôle played by the 24 lines.

Proposition 2.3 *We have* $\mathrm{Pic}\,\overline{V'}=0$, *so that* $\mathrm{Pic}\,\overline{Y}$ *is generated by the classes of the* 24 *lines.*

Proof. It is enough to show that the classes $[l_i]$ and $[s_j]$, $i,j\in\{0,1,2,3\}$, generate $\mathrm{Pic}\,\overline{V}$. The open set U is the complement to a finite set of points in a smooth projective surface, hence $\bar{k}[U]^*=\bar{k}^*$. The same property then holds for V. The spectral sequence $H^p(\mathbf{Z}/2,H^q(\overline{U},\mathbf{G}_m))\Rightarrow H^{p+q}(\overline{V},\mathbf{G}_m)$ gives rise to the exact sequence

$$0\to\mathbf{Z}/2\to\mathrm{Pic}\,\overline{V}\to(\mathrm{Pic}\,\overline{U})^\iota\to 0. \tag{11}$$

(The exactness on the right is due to the fact that $H^2(\mathbf{Z}/2,\bar{k}^*)=0$.) Because of our assumption that $\overline{E}_1$ and $\overline{E}_2$ are not isogenous we have the following isomorphisms of abelian groups:

$$\mathrm{Pic}\,\overline{U}=\mathrm{Pic}\,\overline{D}_1\times\mathrm{Pic}\,\overline{D}_2\simeq E_1(\bar{k})\oplus\mathbf{Z}\oplus E_2(\bar{k})\oplus\mathbf{Z}.$$

Therefore, $(\mathrm{Pic}\,\overline{U})^\iota\simeq E_1[2]\oplus E_2[2]\oplus\mathbf{Z}^2$. The natural map $\mathrm{Pic}\,\overline{V}\to\mathrm{Pic}\,\overline{U}$ is the direct sum of the map $\mathrm{Pic}\,\overline{V}\to\mathrm{Pic}\,\overline{D}_1\simeq E_1(\bar{k})\oplus\mathbf{Z}$ that sends $[l_i]$ to $[i]$, and sends all $[s_j]$ to 0, and the map $\mathrm{Pic}\,\overline{V}\to\mathrm{Pic}\,\overline{D}_2\simeq E_2(\bar{k})\oplus\mathbf{Z}$ that sends $[s_j]$ to $[j]$, and $[l_i]$ to 0. From this description it is clear that the images of the classes $[l_i]$ and $[s_j]$ generate $E_1[2]\oplus E_2[2]\oplus\mathbf{Z}^2=(\mathrm{Pic}\,\overline{U})^\iota$.

It remains to show that the non-trivial element of the kernel of the map $\mathrm{Pic}\,\overline{V}\to\mathrm{Pic}\,\overline{U}$ is a linear combination of the classes $[l_i]$ and $[s_j]$. In $\mathrm{Pic}\,\overline{Y}$ we have

$$-\sum[l_{ij}]=2\sum[l_i]-4[f_1]=2\sum[s_j]-4[f_2].$$

The property $\bar{k}[V]^* = \bar{k}^*$ implies that the kernel of the restriction map $\mathrm{Pic}\,\overline{Y} \to \mathrm{Pic}\,\overline{V}$ is the abelian group $\mathbf{Z}^{16}$ freely generated by the classes of the 16 lines l_{ij}. Hence $\alpha := \sum[l_i] - 2[f_1] \in \mathrm{Pic}\,\overline{V}$ has exact order 2. Due to the fact that in $\mathrm{Pic}\,\overline{V}$ we have $[f_1] = 2[l_i]$ for any i we obtain $\alpha = [l_0] + [l_1] - [l_2] - [l_3]$. On the other hand the inverse image of the divisor $\sum l_i - 2f_1$ in $\mathrm{Pic}\,\overline{U}$ is $\mathrm{div}\,(y_1)$. This completes the proof. QED

This proposition implies that $\mathrm{Pic}\,(Y \times_k L) = \mathrm{Pic}\,\overline{Y}$.

Define the divisor E on $\overline{Y}$ as follows:

$$E = s_0 + s_2 - f_1 - f_2 + l_0 + l_2 + \sum l_{0j} + \sum l_{2j}. \tag{12}$$

Let F be the norm torus $R^1_{L/k}\mathbf{G}_m$. Explicitly F is given by $z_1^2 - az_2^2 = 1$. The module of characters $\widehat{F}$ is the abelian group $\mathbf{Z}$ on which Γ acts through its quotient $\mathrm{Gal}(L/k)$; the non-trivial element of $\mathrm{Gal}(L/k)$ acts as the multiplication by -1. This implies that $H^1(k, \widehat{F}) = \mathbf{Z}/2$. Fix a generator of $\widehat{F}$, and define $\lambda : \widehat{F} \to \mathrm{Pic}\,\overline{Y}$ as the homomorphism which sends this generator to $[E]$. It is clear from (10) that $\mathrm{div}\,(y) = E + \sigma E$, hence λ is a homomorphism of Γ-modules.

By the description of torsors defined by a function whose divisor is a norm ([2], 2.4.2) Y-torsors under F of type λ exist. Any such torsor contains an open subset given by the simultaneous equations (8) and $y = \alpha(z_1^2 - az_2^2) \neq 0$, for some $\alpha \in k^*$. Let $p : Z \to Y$ be the torsor corresponding to $\alpha = 1$.

2.2. Brauer groups of X and Y

Keep the notation and assumptions as above. We start with an almost obvious

Lemma 2.4 $H^1(k, \mathrm{Pic}\,\overline{X}/\mathrm{tors}) = 0$

Proof. Since $f : Y \to X$ is an unramified double covering, we have an exact sequence

$$0 \to \mathbf{Z}/2 \to \mathrm{Pic}\,\overline{X} \to (\mathrm{Pic}\,\overline{Y})^\sigma \to 0,$$

where $(\mathrm{Pic}\,\overline{Y})^\sigma$ is the σ-invariant part of $\mathrm{Pic}\,\overline{Y}$. (This is the exact sequence of low degree terms of the spectral sequence $H^p(\mathbf{Z}/2, H^q(\overline{Y}, \mathbf{G}_m)) \Rightarrow H^{p+q}(\overline{X}, \mathbf{G}_m)$.) Thus we have an isomorphism of Galois modules $\mathrm{Pic}\,\overline{X}/\mathrm{tors} = (\mathrm{Pic}\,\overline{Y})^\sigma$, with the trivial action of the Galois group. Since $H^1(k, \mathbf{Z}) = 0$, the proposition follows. QED

Corollary 2.5 $f^*\mathrm{Br}_1 X = \mathrm{Br}_0 Y$

Proof. We have an injective map $\mathrm{Br}_1 X/\mathrm{Br}_0 X \to H^1(k, \mathrm{Pic}\,\overline{X})$, which is functorial in X. It remains to note that the homomorphism $f^* : H^1(k, \mathrm{Pic}\,\overline{X}) \to H^1(k, \mathrm{Pic}\,\overline{Y})$ factors through $H^1(k, \mathrm{Pic}\,\overline{X}/\mathrm{tors}) = 0$. QED

Proposition 2.6 *(i) The map* $\lambda_* : H^1(k, \widehat{F}) = \mathbf{Z}/2 \to H^1(k, \mathrm{Pic}\,\overline{Y})$ *is an isomorphism.*

(ii) The quaternion algebra $(y, a) \in \mathrm{Br}\,k(Y)$ *is unramified on* Y*. Its class generates* $\mathrm{Br}_1 Y = \mathrm{Br}_\lambda Y$ *modulo* $\mathrm{Br}_0 Y$*. The canonical map* $\mathrm{Br}_1 Y/\mathrm{Br}_0 Y \to H^1(k, \mathrm{Pic}\,\overline{Y})$ *is an isomorphism.*

(iii) Define V_1 *as the complement to the union of* l_{00}*,* l_{01}*,* l_{10}*,* l_{11} *in* Y*. Then the restriction map* $\mathrm{Br}_1 Y \to \mathrm{Br}_1 V_1$ *is an isomorphism.*

Proof. (i) We already saw that $\mathrm{Pic}\,(Y \times_k L) = \mathrm{Pic}\,\overline{Y}$. Thus $H^1(\mathrm{Gal}\,(\bar{k}/L), \mathrm{Pic}\,\overline{Y}) = 0$ and so the inflation map $H^1(\mathrm{Gal}\,(L/k), \mathrm{Pic}\,\overline{Y}) \to H^1(k, \mathrm{Pic}\,\overline{Y})$ is an isomorphism. Note that $\mathrm{Gal}\,(L/k) = \mathbf{Z}/2$ acts on $\mathrm{Pic}\,\overline{Y}$ as σ.

Recall that if M is a $\mathbf{Z}/2$-module, then the Tate cohomology groups of M are 2-periodic, and more precisely, $\hat{H}^{2i}(\mathbf{Z}/2, M)$ is the quotient of the invariants by the norms, and $\hat{H}^{2i+1}(\mathbf{Z}/2, M)$ is the quotient of the anti-invariants by the elements of the form $x - \sigma x$. Our aim is to show that the cohomology class of the anti-invariant element $[E]$ generates $H^1(\mathbf{Z}/2, \mathrm{Pic}\,\overline{Y}) = \mathbf{Z}/2$. Then (i) will follow from the definition of λ.

The singular fibres of π_1 correspond to $x = \pm\sqrt{a}, \pm b\sqrt{a}$. In $\mathrm{Pic}\,\overline{Y}$ each such fibre can be written as $2[l_i] + \sum_j [l_{ij}]$. Let $K = \bar{k}(x)$. The restriction to the generic fibre Y_K gives rise to the exact sequence of $\mathbf{Z}/2$-modules

$$0 \to \mathrm{Vert} \to \mathrm{Pic}\,\overline{Y} \to \mathrm{Pic}\,Y_K \to 0. \tag{13}$$

From the explicit action of σ it is clear that all the fibres of π_1 are split, therefore $H^1(\mathbf{Z}/2, \mathrm{Vert}) = 0$ (in fact, Vert is a permutation module).

Now Y_K is a curve of genus 1 over K. We turn it into an elliptic curve with rational 2-division points by choosing the section s_0 as the origin of the group law. We have an exact sequence

$$0 \to \mathrm{Pic}_0 Y_K \to \mathrm{Pic}\,Y_K \to \mathbf{Z} \to 0. \tag{14}$$

Since $\mathrm{Pic}\,\overline{Y}$ is generated by the classes of the 24 lines (Proposition 2.3) of which all except the s_j are components of the fibres of π_1, we see that $\mathrm{Pic}\,Y_K$ is generated by the restrictions of the $[s_j]$ to the generic fibre Y_K. Hence $\mathrm{Pic}_0 Y_K$ is generated by the differences $[s_j] - [s_{j'}]$, hence $\mathrm{Pic}_0 Y_K \simeq (\mathbf{Z}/2)^2$. (In particular, the rank of Y_K is 0.) We summarize this by rewriting (14) as

$$0 \to (\mathbf{Z}/2)^2 \to \mathrm{Pic}\,Y_K \to \mathbf{Z} \to 0, \tag{15}$$

where $(\mathbf{Z}/2)^2$ is generated by $[s_1] - [s_0]$ and $[s_2] - [s_0]$, and has trivial action of σ.

Let us analyse the sequence (15) with respect to the action of σ. Choose $[s_0]$ as a lifting of the element $1 \in \mathbf{Z}$ to $\mathrm{Pic}\,Y_K$. Then the connecting map $H^0(\mathbf{Z}/2, \mathbf{Z}) \to H^1(\mathbf{Z}/2, (\mathbf{Z}/2)^2) = (\mathbf{Z}/2)^2$ sends 1 to $[s_0] - [s_1]$. This proves that $H^1(\mathbf{Z}/2, \mathrm{Pic}\,Y_K) =$

$\mathbf{Z}/2$, with generator $[s_2]-[s_0]=[s_2]+[s_0]-[f_2]$. (The last equality is due to the fact that in the Picard group of the generic fibre Y_K we have $[f_2]=2[s_j]$.)

Now return to (13) and note that $[E]$ is an anti-invariant lifting of $[s_2]+[s_0]-[f_2]\in \mathrm{Pic}\, Y_K$ to $\mathrm{Pic}\, \overline{Y}$. Hence the non-trivial element of $H^1(\mathbf{Z}/2, \mathrm{Pic}\, Y_K)$ comes from $H^1(\mathbf{Z}/2, \mathrm{Pic}\, \overline{Y})$. This shows that the map $H^1(\mathbf{Z}/2, \mathrm{Pic}\, \overline{Y}) \to H^1(\mathbf{Z}/2, \mathrm{Pic}\, Y_K) = \mathbf{Z}/2$ is an isomorphism, and the non-trivial element of $H^1(\mathbf{Z}/2, \mathrm{Pic}\, \overline{Y})$ is given by $[E]$.

(ii) Using (10) it is straightforward to check that (y,a) is unramified on Y, and hence belongs to $\mathrm{Br}_1 Y$. We show that the image of this element under the canonical map $\mathrm{Br}_1 Y \to H^1(k, \mathrm{Pic}\, \overline{Y})$ is given by $[E]$.

The 2-torsion of the one-dimensional torus F is $\mathbf{Z}/2$. Let $\epsilon : \mathbf{Z}/2 \to F$ be the natural injection. We also have $\widehat{F}/2 = \mathbf{Z}/2$, and the dual surjection $\hat{\epsilon} : \widehat{F} \to \mathbf{Z}/2$. The functoriality of the cup-product implies that $\alpha \cup \epsilon_* \beta = \hat{\epsilon}_*(\alpha) \cup \beta \in \mathrm{Br}\, k(Y)$ for any $\alpha \in H^1(k, \widehat{F})$ and $\beta \in H^1(k(Y), \mathbf{Z}/2)$. Let $\beta = [y] \in k(Y)^*/k(Y)^{*2}$, and let α be the non-trivial element of $H^1(k, \widehat{F})$. It is easy to check that $\hat{\epsilon}_*(\alpha) \in H^1(k, \widehat{F}/2) = k^*/k^{*^2}$ is the class of a. Therefore, (a,y) can be written as the cup-product $\alpha \cup \epsilon_*[y]$. Let $[Z] \in H^1(Y, F)$ be the class of the torsor $p : Z \to Y$ of type λ defined in the end of the previous subsection. The local equation $y = z_1^2 - a z_2^2$ of Z shows that the image of $[Z]$ in $H^1(k(Y), F)$ is $\epsilon_*[y]$. Hence $\alpha \cup \epsilon_*[y]$ is the image in $\mathrm{Br}\, k(Y)$ of $\alpha \cup [Z] \in \mathrm{Br}_1 Y$. The formula of Thm. 4.1.1 of [13] says that the image of $\alpha \cup [Z]$ under the canonical map $\mathrm{Br}_1 Y \to H^1(k, \mathrm{Pic}\, \overline{Y})$ is $\lambda_*(\alpha)$. By the definition of λ this class is given by $[E]$, hence, by (i), it is the non-trivial element of $H^1(k, \mathrm{Pic}\, \overline{Y})$. The proof of (ii) is now complete.

(iii) Let $M \subset \mathrm{Pic}\, \overline{Y}$ be the $\mathbf{Z}/2$-submodule generated by $[l_{00}]$, $[l_{01}]$, $[l_{10}]$, $[l_{11}]$. We have $\bar{k}[V_1]^* = \bar{k}^*$ since the same is true for the smaller open set V. Hence M is freely generated by these four classes. The action of σ is such that it swaps $[l_{00}]$ and $[l_{11}]$, and also $[l_{10}]$ and $[l_{01}]$. Hence M is an induced module, so that $H^1(\mathbf{Z}/2, M) = H^2(\mathbf{Z}/2, M) = 0$.

We claim that $\mathrm{Pic}\, \overline{V}_1$ is torsion-free. Indeed, since $\bar{k}[V]^* = \bar{k}^*$ the kernel of the restriction map $\mathrm{Pic}\, \overline{V}_1 \to \mathrm{Pic}\, \overline{V}$ is freely generated by the 12 remaining classes $[l_{ij}]$. Thus a non-zero torsion element of $\mathrm{Pic}\, \overline{V}_1$ restricts to a non-zero torsion element of $\mathrm{Pic}\, \overline{V}$. It is well-known that every such element comes from $\tau \in \mathrm{Pic}\, \overline{Y}$ such that 2τ is a sum of 8 or 16 of the $[l_{ij}]$ (see [9]; alternatively, this can be checked using the calculations in the proof of Proposition 2.3). So we cannot create a torsion element by removing only four such lines from $\overline{Y}$. We thus have an exact sequence of $\mathbf{Z}/2$-modules

$$0 \to M \to \mathrm{Pic}\, \overline{Y} \to \mathrm{Pic}\, \overline{V}_1 \to 0,$$

which implies that $H^1(\mathbf{Z}/2, \mathrm{Pic}\, \overline{Y}) \to H^1(\mathbf{Z}/2, \mathrm{Pic}\, \overline{V}_1)$ is an isomorphism. Since both modules are torsion-free we conclude that $H^1(k, \mathrm{Pic}\, \overline{Y}) \to H^1(k, \mathrm{Pic}\, \overline{V}_1)$ is also an isomorphism. Now (iii) follows from the last statement of (ii). QED

Now let us turn to the transcendental part of $\operatorname{Br} X$. The rational functions x^2 and t^2 are in the σ-invariant part of $k(Y)$, hence they can be considered as rational functions on X. Let $\mathcal{A} \in \operatorname{Br} k(X)$ be the class of the quaternion algebra

$$((b^2-1)(x^2-a),(c^2-1)(t^2-a)). \tag{16}$$

Proposition 2.7 *The class $\mathcal{A}_Y \in \operatorname{Br} k(Y)$ has the following properties.*

(1) $\mathcal{A}_Y$ is unramified over V_1; it is unramified over Y if and only if either $-d$ or $-ad$ is a square in k^*.

(2) The image of $\mathcal{A}_Y$ in $\operatorname{Br} \bar{k}(Y)$ is unramified.

(3) The image of $\mathcal{A}_Y$ in $\operatorname{Br} \bar{k}(Y)$ is non-zero.

Proof. We prove (1) and (2) at the same time. Let us compute the residues of $\mathcal{A}_Y$. It is clear that $\mathcal{A}_Y$ is unramified away from the 24 lines. (The only thing to check is that the residues at $x=\infty$ and $t=\infty$ are trivial.) We now compute the residues of $\mathcal{A}_Y$ at the 24 lines, that is, the points of codimension 1 that are not in $V' \subset Y$. Each of these residue fields contains L. We note that if $x^2=ab^2$, then $(b^2-1)(x^2-a)$ is a square in L. Similarly, if $t^2=ac^2$, then $(c^2-1)(t^2-a)$ is a square in L. Therefore, all the residues are trivial, except possibly at the points $A=l_0\cup l_1$, $B=s_0\cup s_1$, $C=l_{00}\cup l_{11}$, and $D=l_{01}\cup l_{10}$. It is clear that $\operatorname{res}_A\mathcal{A}_Y=0$ since $\operatorname{val}_A(x^2-a)=2$, whereas t^2-a is a unit. A similar argument shows that $\operatorname{res}_B\mathcal{A}_Y=0$. We have $\operatorname{val}_C(x^2-a)=\operatorname{val}_C(t^2-a)=1$. In order to compute $\operatorname{res}_C\mathcal{A}_Y$ we replace (16) by an equivalent class

$$\begin{aligned}&((b^2-1)(x^2-a),d(c^2-1)(x^2-a)(x^2-ab^2)(t^2-ac^2))=\\&((b^2-1)(x^2-a),-d(b^2-1)(c^2-1)(x^2-ab^2)(t^2-ac^2)).\end{aligned}$$

This shows that $\operatorname{res}_C\mathcal{A}_Y=-d$. By symmetry we also have $\operatorname{res}_D\mathcal{A}_Y=-d$. This proves (1) and (2).

To prove (3) it is enough to show that the restriction of $\mathcal{A}_Y$ to the generic fibre Y_K is a non-zero element of $\operatorname{Br} Y_K$. We think of Y_K as an elliptic curve with rational 2-division points, with s_0 as the origin of the group law. Recall that E_2 is the Jacobian of D_2, so that $\overline{D}_2 \simeq \overline{E}_2$. It is clear from the equation of Y that Y_K is isomorphic to the quadratic twist of the elliptic curve $E_2\times_k K$ by $\rho(x)=(x^2-a)(x^2-ab^2)$. If $Y^2=T(T-p)(T-q)$ is an equation of E_2, then $Y^2=T(T-\rho(x)p)(T-\rho(x)q)$ is an equation of Y_K.

The 2-torsion of the Brauer group of such an elliptic curve is described as follows ([13], Thm. 4.1.1, Example, p. 63, and Exercise 2, p. 91). Every element of $\operatorname{Br} Y_K$ which vanishes at the origin is of the form $(A,T)+(B,T-p)$ for some $A,B\in K^*$. This element is 0 if and only if the class of (A,B) in $(K^*/K^{*2})^2$ is the image of a K-point of Y_K under the Kummer map $Y_K(K)/2Y_K(K)\to H^1(K,(\mathbf{Z}/2)^2)$. Since $Y_K(K)$ consists of 2-division points, we only need to exhibit their images under the

Kummer map. These are $(pq, -\rho(x)p) = (1, \rho(x)), (\rho(x)p, p(p-q)) = (\rho(x), 1)$ and the product of these two elements.

Let us consider the restriction of $\mathcal{A}_Y$ to Y_K. By Tsen's theorem $\mathrm{Br}\, K = 0$, hence any element of $\mathrm{Br}\, Y_K$ vanishes at the origin. Without loss of generality we may assume that $T = (t+\sqrt{a})/(t-\sqrt{a})$. Now our element is given by

$$(x^2 - a, t^2 - a) = (x^2 - a, (t+\sqrt{a})(t-\sqrt{a})^{-1}) = (x^2 - a, T).$$

Since $(x^2 - a, 1) \in (K^*/K^{*2})^2$ is visibly not in the image of $Y_K(K)/2Y_K(K)$ we conclude that the restriction of $\mathcal{A}_Y$ to Y_K is non-zero. This proves (3). QED

The second Betti number of any Enriques surface X equals the rank of $\mathrm{Pic}\, \overline{X}$ (which is 10), and the first Betti number is 0. Thus $\mathrm{Br}\, \overline{X}$ is dual to the torsion subgroup of $\mathrm{Pic}\, \overline{X}$ (see [4], II, Cor. 3.4, III, (8.12)), hence $\mathrm{Br}\, \overline{X} = \mathbf{Z}/2$.

Corollary 2.8 *The image of $\mathcal{A} \in \mathrm{Br}\, k(X)$ in $\mathrm{Br}\, \bar{k}(X)$ is unramified. This image is the unique non-trivial element of $\mathrm{Br}\, \overline{X}$. In particular, the map $f^* : \mathrm{Br}\, \overline{X} \to \mathrm{Br}\, \overline{Y}$ is injective.*

Proof. $\mathcal{A} \otimes \bar{k}$ is obviously unramified away from the images of the 24 lines and the curves given by $x = \infty$ and $t = \infty$. The inverse image of any smooth rational curve in $\overline{X}$ is the disjoint union of two such curves in $\overline{Y}$. Thus if $\mathcal{A} \otimes \bar{k}$ is ramified at the generic point of such a curve on $\overline{X}$, then $\mathcal{A}_Y \otimes \bar{k}$ is also ramified. By symmetry it remains to consider the image of, say $x = \infty$. We note that $t^2 - a$ is a unit, whereas $x^2 - a$ comes from $\bar{k}(\mathbf{P}^1_{\bar{k}})$ via the projection $\overline{X} \to \mathbf{P}^1_{\bar{k}}$. However, the fibre of this map at ∞ is double, hence any function coming from $\bar{k}(\mathbf{P}^1_{\bar{k}})$ has even valuation. Thus the residue is trivial.

Finally, $\mathcal{A} \otimes \bar{k} \neq 0$ since the $\mathcal{A}_Y \otimes \bar{k} \neq 0$ by Proposition 2.7 (3). QED

It seems to be unknown whether the map $f^* : \mathrm{Br}\, \overline{X} \to \mathrm{Br}\, \overline{Y}$ is injective for any Enriques surface X, where $f : Y \to X$ is a K3 covering of X.

Now we are ready to prove the main result of this section.

Theorem 2.9 *Suppose that neither $-d$ nor $-ad$ is a square in k^*. Then $\mathrm{Br}\, X = \mathrm{Br}_1 X$, which implies $f^* \mathrm{Br}\, X = \mathrm{Br}_0 Y$.*

Proof. Let us prove the first statement. Suppose that $\mathcal{B} \in \mathrm{Br}\, X$ is such that $\mathcal{B} \otimes \bar{k} \neq 0$. Since $\mathrm{Br}\, \overline{X} = \mathbf{Z}/2$, from Corollary 2.8 we obtain $\mathcal{B} \otimes \bar{k} = \mathcal{A} \otimes \bar{k}$. Let $\mathcal{B}_Y = f^* \mathcal{B}$. By Proposition 2.7 (1) $\mathcal{B}_Y - \mathcal{A}_Y$ is unramified on V_1, hence belongs to $\mathrm{Br}_1 V_1 = \mathrm{Br}_1 Y$ (Proposition 2.6 (*iii*)). Thus $\mathcal{A}_Y - \mathcal{B}_Y$ is unramified, hence $\mathcal{A}_Y$ is also unramified. This contradicts Proposition 2.7 (1). Thus $\mathcal{B} \in \mathrm{Br}_1 X$. We have proved that $\mathrm{Br}\, X = \mathrm{Br}_1 X$. The second statement now follows from Corollary 2.5. QED

2.3. Counter-example to weak approximation not explained by the Manin obstruction

Let $k = \mathbf{Q}$ be the field of rational numbers. Let a and $b = p$ be primes such that a is 1 modulo 4, and a is not a square modulo p. Let c be an integer such that $c(c^2 - 1)$ is not divisible by p. Consider the Kummer surface Y over $\mathbf{Q}$ given by the affine equation

$$y^2 = (x^2 - a)(x^2 - ap^2)(t^2 - a)(t^2 - ac^2), \tag{17}$$

and the corresponding Enriques surface $X = Y/\sigma$.

If we choose $a = 5$, $b = 13$, $c = 2$ as in the Example after Lemma 2.2, then the above conditions are satisfied. The elliptic curves E_1 and E_2 are not isogenous over $\overline{\mathbf{Q}}$, so that all the computations of the previous subsection are valid. Moreover, the set $X(\mathbf{Q})$ is Zariski dense in X.

We now construct a family of local points on X. By substituting $x = t = p^{-1}$ into (17) we obtain $y^2 = p^{-8}\alpha^2$, where α is a p-adic unit congruent to 1 modulo p. Let N_p be the $\mathbf{Q}_p$-point on Y with coordinates $x = t = p^{-1}$, $y = p^{-4}\alpha$. Consider the $\mathbf{Q}$-point M on Y with coordinates $x = t = 0$, $y = a^2pc$. For any prime $q \neq p$ we define $N_q = M$, and we do likewise for the archimedean place. We obtain an adelic point $\{N_v\}$ on Y.

Theorem 2.10 *The adelic point $\{f(N_v)\}$ is in $X(\mathbb{A}_{\mathbf{Q}})^{\mathrm{Br}}$ but not in the closure of $X(\mathbf{Q})$. Hence X is a counter-example to weak approximation that is not accounted for by the Brauer–Manin obstruction.*

Proof. In our previous notation $d = 1$. Since -1 and $-a$ are not squares in $\mathbf{Q}^*$, Theorem 2.9 applies. Since $f^*\mathrm{Br}\, X = \mathrm{Br}\, \mathbf{Q}$ the first statement immediately follows from the projection formula.

By the global reciprocity we have $\sum_{v\in\Omega_{\mathbf{Q}}} \mathrm{inv}_v(a^2pc, a) = 0$. On the other hand, $\mathrm{inv}_p(a^2pc, a) \neq 0$ since a is not a square modulo p. We have $\mathrm{inv}_p(p^{-4}\alpha, a) = 0$ since α and a are p-adic units. It follows that the adelic point $\{N_v\} \in Y(\mathbb{A}_{\mathbf{Q}})$ does not satisfy the Brauer–Manin condition with respect to the Azumaya algebra (y, a).

Recall that $p : Z \to Y$ is a Y-torsor of type λ defined in the end of Subsection 2.1. We now compose the torsors $p : Z \to Y$ and $f : Y \to X$. Indeed, all the conditions of Proposition 1.4 are satisfied. (The image of λ is H-invariant, as it is generated by the σ-anti-invariant element $[E]$.) We obtain an X-torsor $g : Z \to X$ under a k-group G; this group is an extension

$$1 \to F \to G \to \mathbf{Z}/2 \to 1.$$

The class of (y, a) is in $\mathrm{Br}_\lambda Y$ by Proposition 2.6 (*ii*). By the descent theory ([13], Thm. 6.1.2) the fact that the adelic point $\{N_v\} \in Y(\mathbb{A}_{\mathbf{Q}})$ does not satisfy the

Brauer–Manin condition given by an element of $\mathrm{Br}_{\lambda} Y$ implies that $\{N_v\} \notin Y(\mathbb{A}_{\mathbf{Q}})^p$. The closure of $X(\mathbf{Q})$ in $X(\mathbb{A}_{\mathbf{Q}})$ is contained in $X(\mathbb{A}_{\mathbf{Q}})^g$, thus to prove the theorem it is enough to prove the following

Proposition 2.11 *The adelic point $\{P_v\} = \{f(N_v)\}$ is not contained in the set $X(\mathbb{A}_{\mathbf{Q}})^g$. There is a non-abelian descent obstruction to weak approximation on X for $\{P_v\}$.*

Proof. We have an exact sequence of pointed sets

$$\mathbf{Z}/2 \to H^1(\mathbf{Q}, F) \to H^1(\mathbf{Q}, G) \to H^1(\mathbf{Q}, \mathbf{Z}/2)$$

Let us compute the image of the non-trivial element $h \in \mathbf{Z}/2$ under the connecting map $\delta : \mathbf{Z}/2 \to H^1(\mathbf{Q}, F) = \mathbf{Q}^*/N_{L/\mathbf{Q}}(L^*)$. If φ_h is a lifting of h to $G(\overline{\mathbf{Q}})$, then $\delta(h)$ is the class of the cocycle $\sigma(\gamma) = \varphi_h^{-1} \cdot {}^{\gamma}(\varphi_h)$ (see [10], I.5.4). We obtain (cf. (5))

$${}^{\gamma}(\varphi_h(z)) = \varphi_h(\sigma(\gamma) \cdot {}^{\gamma}z), \qquad z \in Z(\overline{\mathbf{Q}}), \quad \gamma \in \Gamma.$$

Let Z^{σ} be the twisted torsor of Z by σ. The displayed formula shows that φ_h is an isomorphism of $\mathbf{Q}$-varieties $Z^{\sigma} \to Z$. We also have $\varphi_h(tz) = \tau_h(t)\varphi_h(z)$ for any $t \in F(\overline{\mathbf{Q}})$, $z \in Z(\overline{\mathbf{Q}})$, where τ_h is the natural action of $H(\overline{\mathbf{Q}})$ on $F(\overline{\mathbf{Q}})$ (as the proof of Proposition 1.4; note that in our case $\tau_h(t) = t^{-1}$). This shows that we actually have an isomorphism of Y-torsors $h^*(Z)^{\sigma} \to h_*(Z)$. Therefore $[h^*(Z)] - [\sigma] = [h_*(Z)]$, so that $\delta(h) = [\sigma] = [h^*(Z)] - [h_*(Z)]$. To compute this difference we can restrict the classes to $H^1(\mathbf{Q}(Y), F)$. On the one hand, the local equation $y = z_1^2 - a z_2^2$ of Z shows that $h^*[y] = [y] + [-1]$. On the other hand, the map $\tau_h(t) = t^{-1}$ induces the trivial action on $H^1(k(Y), F)$, since the latter group is 2-torsion. Hence $h_*[y] = [y]$. Putting all this together we conclude that $\delta(h)$ is the class of -1 in $\mathbf{Q}^*/N_{L/\mathbf{Q}}(L^*)$.

Since a is a prime which is 1 mod 4, -1 is the norm of an element of $L = \mathbf{Q}(\sqrt{a})$. Thus δ is trivial. The same is of course true if the ground field $k = \mathbf{Q}$ is replaced by any bigger field. We obtain a commutative diagram of pointed sets with exact rows

$$\begin{array}{ccccccc}
 & & H^1(\mathbf{Q}, F) & \longrightarrow & H^1(\mathbf{Q}, G) & \longrightarrow & H^1(\mathbf{Q}, \mathbf{Z}/2) \\
 & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & \prod_{v \in \Omega_{\mathbf{Q}}} H^1(\mathbf{Q}_v, F) & \longrightarrow & \prod_{v \in \Omega_{\mathbf{Q}}} H^1(\mathbf{Q}_v, G) & \longrightarrow & \prod_{v \in \Omega_{\mathbf{Q}}} H^1(\mathbf{Q}_v, \mathbf{Z}/2)
\end{array}$$

Since $H^1(\mathbf{Q}_v, F)$ is either zero or $\mathbf{Z}/2$, the map $H^1(\mathbf{Q}_v, F) \to H^1(\mathbf{Q}_v, G)$ is injective for any place v. The diagonal map $H^1(\mathbf{Q}, \mathbf{Z}/2) \to \prod_{v \in \Omega_{\mathbf{Q}}} H^1(\mathbf{Q}_v, \mathbf{Z}/2)$ is obviously injective.

Suppose that $\{P_v\} \in X(\mathbb{A}_{\mathbf{Q}})^g$. Set $g_v = [g^{-1}(P_v)] \in H^1(k_v, G)$. Then by definition $\{g_v\}$ is in the diagonal image of $H^1(\mathbf{Q}, G)$ in $\prod_{v \in \Omega_{\mathbf{Q}}} H^1(\mathbf{Q}_v, G)$. Since g_v is the image of $f_v = [p^{-1}(N_v)]$, the injectivity of the map $H^1(k_v, F) \to H^1(k_v, G)$ implies by an easy diagram chase that $\{f_v\}$ is in the diagonal image of $H^1(k,F)$. But this is not possible because $\{N_v\}$ does not belong to $Y(\mathbb{A}_{\mathbf{Q}})^p$. QED

For the sake of completeness let us also give an alternative argument that $\{f(N_v)\}$ cannot be approximated by a rational point (cf. [12]).

Since (y,a) is unramified on Y there exists a finite set of places S such that for $v \notin S$ the local invariant of (y,a) at any $\mathbf{Q}_v$-point on Y is 0. The involution σ sends (y,a) to $(-y,a)$. The quaternion algebra $(-1,a)$ is trivial because the prime a is congruent ot 1 modulo 4, and so is a norm for $\mathbf{Q}(\sqrt{-1})/\mathbf{Q}$. Thus (y,a) is σ-invariant.

Suppose that $\{f(N_v)\}$ is in the closure of $X(\mathbf{Q})$. Since $f: Y \to X$ is unramified, there exists a finite set of quadratic fields $k_1,\ldots,k_n$ with the property that for any $P \in X(\mathbf{Q})$ the residue field of a closed point of Y over P is either $\mathbf{Q}$ or one of the k_i. Let p_i be a prime that is inert in k_i. Suppose that $R \in X(\mathbf{Q})$ is close enough to $f(N_v)$ in the $\mathbf{Q}_v$-topology for all $v \in S \cup \{p_1,\ldots,p_n\}$. If $\mathbf{Q}(f^{-1}(P))$ is a quadratic field, then it must be split at all the primes p_i. This is a contradiction. Therefore, the inverse image of R in Y must consist of two $\mathbf{Q}$-points, say R_1 and R_2. By the implicit function theorem for the local field $\mathbf{Q}_v$ we know that N_v is very close to either R_1 or R_2. But (y,a) is σ-invariant, hence for any place v we have $\mathrm{inv}_v((y,a)(R_1)) = \mathrm{inv}_v((y,a)(R_2))$. Since the local invariant is locally constant we see that $\mathrm{inv}_v((y,a)(N_v)) = \mathrm{inv}_v((y,a)(R_1))$ for all $v \in S \cup \{p_1,\ldots,p_n\}$. Since $\mathrm{inv}_v((y,a)(N_v)) = 0$ for $v \notin S$, we obtain

$$\sum_{v\in\Omega_{\mathbf{Q}}} \mathrm{inv}_v((y,a)(N_v)) = \sum_{v\in\Omega_{\mathbf{Q}}} \mathrm{inv}_v((y,a)(R_1)) = 0$$

by the global reciprocity. But this sum is non-zero, as we showed in the beginning of proof of Theorem 2.10.

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