Algebra IV

Test 2

30 November 2017

1. Let k be a field. For a positive integer n let R be the ring of $(n \times n)$-matrices with entries in k, with usual addition and multiplication of matrices. Let $M = k^n$ be the vector space of column vectors. Turn M into a left R-module by letting a matrix $A \in R$ act on the column vector v by the usual matrix multiplication: A applied to v is Av.

Determine $\mathrm{Hom}_R(M, M)$.

Solution (5 marks, if full details are given)

R contains k, embedded into R as the subring of scalar matrices xI_n, where I_n is the identity matrix and $x \in k$. Hence an R-module is also a k-module, i.e. a vector space over k. Thus any map of R-modules $f : M \to M$ is a linear transformation so $f(v) = Av$, where A is a $(n \times n)$-matrix $A \in R$. It is compatible with the action of R on M if and only if $f(Bv) = Bf(v)$, which is $AB = BA$, for any $B \in R$. This implies that $A = xI_n$ for some $x \in k$. Hence $\mathrm{Hom}_R(M, M) = k$.

2. Construct an injective resolution of the abelian group $\mathbb{Z} \oplus \mathbb{Z}/2$.

Solution (6 marks, with less marks for partially correct or incomplete solutions)

$\mathbb{Q}$ is divisible, hence injective, and $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is the tautological embedding. Since $\mathbb{Q}/\mathbb{Z}$ is also divisible, we see that $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ is an injective resolution of $\mathbb{Z}$. Next, consider the multiplication by 2 map $\mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ of injective groups. Its kernel is isomorphic to $\mathbb{Z}/2$, so $\mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \to 0$ is an injective resolution of $\mathbb{Z}/2$. Taking direct sums we obtain the following resolution of $\mathbb{Z} \oplus \mathbb{Z}/2$ (where the maps are the direct sums of the above maps):

$$\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z} \longrightarrow \mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

(This resolution is not unique. Any correct resolution is fine.)

3. Let m and n be positive integers. Let $\mu : \mathbb{Z}/n \to (\mathbb{Z}/n)^m$ be the map sending 1 to the all-one vector $(1, \ldots, 1)$. Let $\nu : (\mathbb{Z}/n)^m \to \mathbb{Z}/n$ be the map sending $(x_1, \ldots, x_m)$, where $x_i \in \mathbb{Z}/n$, to the sum $x_1 + \ldots + x_m$.

(a) Determine when

$$\mathbb{Z}/n \xrightarrow{\mu} (\mathbb{Z}/n)^m \xrightarrow{\nu} \mathbb{Z}/n$$

is a complex.

(b) When this is a complex, compute its (middle) homology group.

Solution (4=2+2 marks)

(a) It is a complex if and only if $\nu\mu = 0$, i.e. m is zero in $\mathbb{Z}/n$, which means that n divides m.

(b) We have $\mathrm{Ker}(\nu) \cong (\mathbb{Z}/n)^{m-1}$, $\mathrm{Im}(\mu) \cong \mathbb{Z}/n$, hence the homology group is $(\mathbb{Z}/n)^{m-2}$.

4. Compute $\mathbb{R}^* \otimes_{\mathbb{Z}} \mathbb{Z}/2$. (Here $\mathbb{R}^*$ is the multiplicative group of $\mathbb{R}$.)

Solution (5 marks)

By lectures, for any abelian group A we have $A \otimes_{\mathbb{Z}} \mathbb{Z}/2 \cong A/2$. Hence $\mathbb{R}^* \otimes_{\mathbb{Z}} \mathbb{Z}/2$ is canonically isomorphic to the quotient of $\mathbb{R}^*$ by the subgroup of squares, that is, by $\mathbb{R}_{>0}$. The abelian group $\mathbb{R}^*/\mathbb{R}_{>0}$ is cyclic of order 2, hence $\mathbb{R}^* \otimes_{\mathbb{Z}} \mathbb{Z}/2 \cong \mathbb{Z}/2$.