Algebra IV

Problem Sheet 1

October 17, 2017

1. Construct infinitely many pairwise non-isomorphic projective $\mathbb{Z}/12$-modules, neither of which is a free $\mathbb{Z}/12$-module.

2. (a) When is $0 \to A \to B \to 0$ a complex? When is this an exact sequence?

(b) Prove that a short exact sequence of left R-modules

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

is split if and only if there is a map $\rho : B \to A$ such that $\rho\alpha = \mathrm{id}_A$. (Such a map is called a *retraction* of α.)

3. Let k be a field. Recall that k-modules and vector spaces over k are the same objects. Prove that any finite dimensional vector space over k is a free k-module. Deduce that any short exact sequence of finite dimensional vector spaces is split. Now show that any finite dimensional vector space over k is an injective k-module.

4. Show that $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ is an exact sequence of abelian groups. Is it split? Is $\mathbb{Z}$ injective as an abelian group? Justify your answers.

5. For an abelian group M we write $M[n] = \{x \in M | nx = 0\}$. Let M/n denote the quotient of M by its subgroup $nM = \{nx | x \in M\}$. Prove that if

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

is an exact sequence of abelian groups, then the following sequence is exact:

$$0 \to A[n] \to B[n] \to C[n] \xrightarrow{\gamma} A/n \to B/n \to C/n \to 0 \qquad (1)$$

Here the maps are induced by α or β, except the connecting map γ, which is defined as follows. Let $c \in C$ be such that $nc = 0$. There exists a $b \in B$ such that $\beta(b) = c$. Thus $\beta(nb) = nc = 0$, so $nb \in A$. Then $\gamma(c)$ is defined as the coset of nb in A/n. [Hint: You need to show that γ is well defined, i.e. $\gamma(c)$ will not change if you choose a different element of B that goes to c. Then show that (1) is a complex, and then prove that it is exact at each term. If you know the snake lemma, you can use it to get a slick proof of the exactness of (1) in no time at all.]