On the equations for universal torsors over del Pezzo surfaces

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à Jean-Louis Colliot-Thélène

Introduction

Universal torsors were invented by Jean-Louis Colliot-Thélène and Jean-Jacques Sansuc; for smooth projective varieties $X$ with $H^1(X, O) = 0$ they play the role similar to that of $n$-coverings of elliptic curves. The foundations of the theory of descent on torsors were laid in a series of notes in Comptes Rendus de l’Académie des Sciences de Paris in the second half of the 1970’s, and a detailed account was published in [4]. The theory has strong number theoretic applications if the torsors can be described by explicit equations, and if the resulting system of equations can be treated using some other methods, whether algebraic or analytic. Such is the case for surfaces fibred into conics over $\mathbb{P}^1$: the universal torsors over these surfaces are closely related to complete intersections of quadrics of a rather special kind. To describe them we use the following terminology. If $Z \subset \mathbb{A}^m_k$ is a closed subset of an affine space with a coordinate system over a field $k$, then the variety obtained from $Z$ by multiplying coordinates by non-zero numbers will be called a dilatation of $Z$. If exactly $n$ geometric fibres of the conic bundle $X \to \mathbb{P}^1$ are singular, then there is a non-degenerate quadric $Q \subset \mathbb{A}^{2n}_k$ such that the universal torsors over $X$ are stably birationally equivalent to the product of a complete intersection of $n-2$ dilatations of $Q$, and a Severi–Brauer variety (see [4], Thm. 2.6.1). This description was key for a plethora of applications of the descent theory to the Hasse principle, weak approximation, zero-cycles, R-equivalence and rationality problems (the Zariski conjecture), see, e.g., [6] and [2]. A stumbling block for a similar treatment of cubic and more general smooth del Pezzo surfaces without a pencil of rational curves is, possibly, the absence of a satisfactory presentation of their universal torsors. Known descriptions of universal torsors over diagonal cubic surfaces ([4], 2.5, [5], 10) lack the simplicity and the symmetry manifest in the conic bundle case.

One way to look at a non-degenerate quadric in $\mathbb{A}^{2n}_k$ is to think of it as a homogeneous space of the semisimple Lie group $G$ associated with the root system $D_n$. 
which naturally appears in connection with conic bundles with \( n \) singular fibres, see e.g. [9]. Indeed, over an algebraically closed field we can identify \( Q \) with the orbit of the highest weight vector of the fundamental \( 2n \)-dimensional representation \( V \) of \( G \). Then an ‘essential part’ of the torsor is the intersection of \( n - 2 \) dilatations of this homogeneous space by the elements of a maximal torus in \( \text{GL}(V) \).

The aim of this paper is to generalize this description from the case of conic bundles to that of del Pezzo surfaces. (Recall that these two families exhaust all minimal smooth projective rational surfaces, according to the classification of Enriques–Manin–Iskovskih.) We build on the results of our previous paper [13], where we studied split del Pezzo surfaces, i.e. the case when the Galois action on the set of exceptional curves is trivial. The main result of [13] is a construction of an embedding of a universal torsor over a split del Pezzo surface \( X \) of degree 5, 4, 3 or 2 into the orbit of the highest weight vector of a fundamental representation of the semisimple simply connected Lie group \( G \) which has the same root system as \( X \), i.e. \( A_4, D_5, E_6 \) or \( E_7 \). This orbit is the punctured affine cone over \( G/P \), where \( P \subset G \) is a maximal parabolic subgroup. The embedding is equivariant with respect to the action of the Néron–Severi torus \( T \) of \( X \), identified with a split maximal torus of \( G \) extended by \( \mathbb{G}_m \). In Theorem 2.5 we describe universal torsors over split del Pezzo surfaces of degree \( d \) as intersections of \( 6 - d \) dilatations of the affine cone over \( G/P \) by \( k \)-points of a maximal torus of \( \text{GL}(V) \) which is the centralizer of \( T \subset \text{GL}(V) \). This gives a more conceptual approach to the equations appeared previously in the work of Popov [11] and Derenthal [7]. This approach can be called a global description of torsors compared to their local description obtained by Colliot-Thélène and Sansuc in [4], 2.3.

For a general del Pezzo surface \( X \) of degree 4, 3 or 2 with a rational point we construct an embedding of a universal torsor over \( X \) into the same homogeneous space as in the split case, but this time equivariantly with respect to the action of a (possibly, non-split) maximal torus of \( G \), see Theorem 4.4. The case of del Pezzo surfaces of degree 5, where a rational point comes for free by a theorem of Enriques and Swinnerton-Dyer, was already known ([15], Thm. 3.1.4). The proof of Theorem 4.4 uses a recent result of Philippe Gille [8] and M.S. Raghunathan [12] which describes possible Galois actions on the character group of a maximal torus in a quasi-split algebraic group. This result implies that the Néron–Severi torus \( T \) of \( X \) embeds into the same split group \( G \) extended by \( \mathbb{G}_m \), exactly as in the case of a split del Pezzo surface.

The condition on the existence of a rational point on \( X \) is not a restriction in the case of degree 5, but is clearly a restriction for smaller degrees, limiting the scope of possible applications. However, if \( X \) is a del Pezzo surface of degree 4, this condition is necessary as well as sufficient for our construction: if \( X \) can be realized inside a twisted form of the quotient of \( G/P \) by a maximal torus, then \( X \) has a rational point, see Corollary 4.5 (i). Finally, in Corollary 4.5 (ii) we show that any
del Pezzo surface of degree 4 with a k-point has a universal torsor which is a dense open subset of the intersection of the affine cone over $G/P$ with its dilatation by a k-point of the centralizer of $T$ in $GL(V)$.

We recall the construction of [13] in Section 1 alongside with all necessary notation. In Section 2 we describe torsors over split del Pezzo surfaces as intersections of dilatations of the affine cone over $G/P$. In Section 3 we prove a uniqueness property used in the proof of the main results in the non-split case in Section 4.

The ideas developed in this paper originate in the second author’s discussions with Victor Batyrev, to whom we are deeply grateful.

1 Review of the split case

Preliminary remarks Let $k$ be a field of characteristic 0 with an algebraic closure $\overline{k}$.

Let $V$ be a vector space over $k$, and let $T \subset GL(V)$ be a split torus, i.e. $T \simeq G^n_m$ for some $n$. Let $\Lambda \subset \hat{T}$ be the set of weights of $T$ in $V$, and let $V_\lambda \subset V$ be the subspace of weight $\lambda$. We have $V = \oplus_{\lambda \in \Lambda} V_\lambda$. Let $S$ be the centralizer of $T$ in $GL(V)$, i.e.

$$S = \prod_{\lambda \in \Lambda} GL(V_\lambda) \subset GL(V).$$

In what follows we always assume that $\dim V_\lambda = 1$ for all $\lambda \in \Lambda$; then $S$ is a maximal torus in $GL(V)$. Let $\pi_\lambda : V \to V_\lambda$ be the natural projection. For $A \subset V$ we write $A^\times$ for the set of points of $A$ outside $\cup \pi^{-1}_\lambda(0)$.

Let $r = 4, 5, 6$ or 7. A split del Pezzo surface $X$ of degree $d = 9 - r$ is the blowing-up of $\mathbb{P}^2$ in $r$ k-points in general position (i.e., no three points are on a line and no six are on a conic). The Picard group $\text{Pic} X$ is a free abelian group of rank $r + 1$, generated by the classes of exceptional curves on $X$. Let $T = G^*_m+1$. Once an isomorphism $\hat{T} \to \text{Pic} X$ is fixed, $T$ is called the Néron–Severi torus of $X$. A universal torsor $f : T \to X$ is an $X$-torsor with structure group $T$, whose type is the isomorphism $T \simeq \text{Pic} X$ (see [15], p. 25). We call a divisor in $\mathcal{T}$ an exceptional divisor if it is the inverse image of an exceptional curve in $X$.

Now suppose that $\dim V$ equals the number of exceptional curves on $X$. We can make an obvious but useful observation.

**Lemma 1.1** Let $\mathcal{T} \to X$ be a universal torsor over a split del Pezzo surface $X$. Let $\phi$ and $\psi$ be $T$-equivariant embeddings $\mathcal{T} \to V$ such that for each weight $\lambda \in \Lambda$ the divisors of functions $\pi_\lambda \phi$ and $\pi_\lambda \psi$ are equal to the same exceptional divisor with multiplicity 1. Then $\psi = s \circ \phi$ for some $s \in S(k)$.

**Proof** Since $\mathcal{T}$ is a universal torsor we have $k[\mathcal{T}]^* = k^*$, hence two regular functions with equal divisors differ by a non-zero multiplicative constant. QED
Construction in the split case Let the pair consisting of a root system $R$ of rank $r$ and a simple root $\alpha$ be one of the pairs in the list

\[(A_4, \alpha_3), \ (D_5, \alpha_5), \ (E_6, \alpha_6), \ (E_7, \alpha_7). \quad (1)\]

Here and elsewhere in this paper we enumerate roots as in [3]. Let $G$ be the split simply connected simple group with split maximal torus $H$ and root system $R$. Let $\omega$ be the fundamental weight dual to $\alpha$, and let $V = V(\omega)$ be the irreducible $G$-module with the highest weight $\omega$. It is known that $V$ is faithful and minuscule, see [3]. Let $P \subset G$ be the maximal parabolic subgroup such that $G/P \subset \mathbb{P}(V)$ is the orbit of the highest weight vector. The affine cone over $G/P$ is denoted by $(G/P)_a$.

It is easy to check that the $G$-module $S^2(V)$ is the direct sum of two irreducible submodules $V(\omega_1) \oplus V(2\omega)$. For $r \leq 6$, $V(\omega_1)$ is a non-trivial irreducible $G$-module of least dimension; it is a minuscule representation of $G$. If $r = 7$, then $V(\omega_1)$ is the adjoint representation; it is quasi-minuscule, that is, all the non-zero weights have multiplicity 1 and form one orbit of $W$. If $pr$ is the natural projection $V \to V(\omega_1)$, and $\text{Ver} : V \to S^2(V)$ is the Veronese map $x \mapsto x^2$, then it is well known that $(G/P)_a = (pr \circ \text{Ver})^{-1}(0)$ (see [1], Prop. 4.2 and references there).

Since the eigenspaces of $H$ in $V$ are 1-dimensional, $V$ has a natural coordinate system with respect to which $S$ is the ‘diagonal’ torus. Let the torus $T \subset S$ be the extension of $H$ by the scalar matrices $G_m \subset \text{GL}(V)$. Note that an eigenspace of $H$ in $V$ is also an eigenspace of $T$, so that there is a natural bijection between the corresponding sets of characters.

Let $V^{sf}$ be the dense open subset of $V$ consisting of the points whose $H$-orbits are closed and whose stabilizers in $T$ are trivial. Let $(G/P)_a^{sf} = (G/P)_a \cap V^{sf}$. In [13] we constructed a $T$-equivariant closed embedding of $T$ into $(G/P)_a^{sf}$ such that each weight hyperplane section $T \cap \pi^{-1}(0)$ is an exceptional divisor with multiplicity 1. Then $X^\times = f(T^\times)$ is the complement to the union of exceptional curves on $X$.

We need to recall the details of this construction. It starts with the case $(R, \alpha) = (A_4, \alpha_3)$ where the torsor $T$ is the set of stable points of $(G/P)_a$ which is the affine cone over the Grassmannian $\text{Gr}(2,5)$, see [15], 3.1. Thus $T$ is open and dense in $(G/P)_a$ in this case. As in [13] we use dashes to denote the previous pair in (1); the previous pair of $(A_4, \alpha_3)$ is $(A_1 \times A_2, \alpha_1^{(1)} + \alpha_2^{(2)})$, though it will not be used. For $r \geq 5$ we assume that a torsor $T' \subset (G'/P')_a^{sf}$ over a split del Pezzo surface of degree $10 - r$ is already constructed, and proceed to construct $T$ as follows.

Let $\Lambda_n \subset \Lambda$ be the set of weights $\lambda$ such that $n$ is the coefficient of $\alpha$ in the decomposition of $\omega - \lambda$ into a linear combination of simple roots. Let $V_n = \bigoplus_{\lambda \in \Lambda_n} V_\lambda$, then

\[ V = \bigoplus_{n \geq 0} V_n. \quad (2) \]

The subspaces $V_n$ are $G'$-invariant. In fact, $V_n = 0$ for $n > 3$ so that

\[ V = V_0 \oplus V_1 \oplus V_2 \oplus V_3, \]
and $V_3 = 0$ unless $r = 7$. The degree 0 component $V_0 \simeq k$ is the highest weight subspace, and the degree 1 component $V_1$ is isomorphic to $V'$ as a $G'$-module. The $G'$-module $V_2$ is irreducible with highest weight $\omega_1$. For $r = 7$ we have $V_3 \simeq k$. 

Recall from [13] that $g_t = (t, 1, t^{-1}, t^{-2})$ is an element of $T$, for any $t \in \overline{k}^*$. Let $U \subset (G/P)_a$ be the set of points of $(G/P)_a$ outside $(V_0 \oplus V_1) \cup (V_2 \oplus V_3)$. The natural projection $\pi : V \to V_1$ defines a morphism $U \to V_1 \setminus \{0\}$ which is the composition of a torus under $G_m = \{g_t | t \in \overline{k}^*\}$ and the morphism inverse to the blowing-up of $(G'/P')_a \setminus \{0\}$ in $V_1 \setminus \{0\}$ ([13], Cor. 4.2). There is a $G'$-equivariant affine morphism $\exp : V_1 \to (G/P)_a$ such that $\pi \circ \exp = \text{id}$, and the affine cone over $\exp(V_1)$ is dense in $(G/P)_a$. As in [13] we write $\exp(x) = (1, x, p(x), q(x))$. The map $p$ can be identified, up to a non-zero constant, with the composition of the second Veronese map and the natural projection $S^2(V_1) \to V_2$, so that $(G'/P')_a = p^{-1}(0)$. Since $V_2$ is the direct sum of 1-dimensional weight spaces, it has a natural coordinate system. The weight coordinates of $p(x)$ will be written as $p_\mu(x)$, where $\mu \in W\omega_1$.

The choice of a point in $V^\times$ defines an isomorphism $V^\times \simeq S$ compatible with the action of $S$. Using this isomorphism we define a multiplication on $V^\times$, and then extend it to $V$. However, none of our formulae depend on this isomorphism.

Suppose that $\mathcal{T} \subset (G'/P')_a \subset V = V_1$ is such that $f' : \mathcal{T}' \to X' = \mathcal{T}' / T'$ is a universal torsor over a del Pezzo surface $X'$, moreover, the $T'$-invariant hyperplane sections of $\mathcal{T}'$ are the exceptional divisors. In [13] we proved that for any $\overline{k}$-point $x_0$ in $\mathcal{T}^{\times}$ there exists a non-empty open subset $\Omega(x_0) \subset (G'/P')_a^\times$, whose definition is recalled in the beginning of the next section, such that for any $y_0$ in $\Omega(x_0)$ the orbit $T'y_0$ is the scheme-theoretic intersection $x_0^{-1}y_0T' \cap (G'/P')_a$ (see Cor. 6.5 of [13]). Therefore, if $\mathcal{T}$ is the proper transform of $x_0^{-1}y_0T'$ in $U$, then $X = \mathcal{T} / T'$ is the blowing-up of $X'$ at the image of $x_0$. Consequently one proves that $\mathcal{T} \subset (G/P)_a^{sf}$. Equivalently, $\mathcal{T}$ can be defined as the affine cone (without zero) over the Zariski closure of $\exp(x_0^{-1}y_0T' \setminus T'y_0)$ in $(G/P)_a^{sf}$.

The construction of an embedding of a universal torsor over $X$ into $(G/P)_a^{sf}$ is a main result of [13] (Thm. 6.1). The following corollary to this theorem complements it by showing that our embedding is in a sense unique.

**Corollary 1.2** Let $\mathcal{T} \subset V^{sf}$ be a closed $T$-invariant subvariety such that $\mathcal{T} / T$ is a split del Pezzo surface and the weight hyperplane sections of $\mathcal{T}$ are exceptional divisors with multiplicity 1. Then for some $s \in S(k)$ the torsor $s\mathcal{T}$ is a subset of $(G/P)_a$ obtained by our construction (for some choice of a basis of simple roots of our root system $R$).

**Proof** The construction of [13] recalled above produces a universal torsor $\overline{T}$ over the same split del Pezzo surface $X$ inside $(G/P)_a$, satisfying the condition that the weight hyperplane sections are the exceptional divisors with multiplicity 1. The identifications of the exceptional curves on $X$ with the weights of $V$ coming from
and $\tilde{T}$ may be different, however the permutation that links them is an automorphism of the incidence graph of the exceptional curves on $X$. It is well known (see [10]) that the group of automorphisms of this graph is the Weyl group $W$ of $R$. Thus replacing $\tilde{T}$ by its image under the action of an appropriate element of $W$ (that is, a representative of this element in the normalizer of $H$ in $G$), we ensure that the identification of the weights with the exceptional curves is the same for both embeddings. (The choice of this element in $W$ is equivalent to the choice of a basis of simple roots in our construction.) The multiplicity 1 condition in the construction of [13] is easily checked by induction from the case $r = 4$ where we consider the Plücker coordinate hyperplane sections of $Gr(2, 5)$. It remains to apply Lemma 1.1. QED

Let us recall some more notation. For $\mu \in W\omega_1 \subset \hat{H}$ we write $S^2_\mu(V)$ for the $H$-eigenspace of $S^2(V)$ of weight $\mu$, and $S^3_\mu(V)^*$ for the dual space. Let $\text{Ver}_\mu$ be the Veronese map $V \to S^2(V)$ followed by the projection to $S^2_\mu(V)$. For $r = 6$ we write $S^3_0(V)$ for the zero weight $H$-eigenspace in $S^3(V)$, and $\text{Ver}_0 : V \to S^3_0(V)$ for the corresponding natural map.

As in the previous corollary, we denote by $T \subset V^{sf}$ a closed $T$-invariant subvariety such that $T/T$ is a split del Pezzo surface and the weight hyperplane sections of $T$ are exceptional divisors with multiplicity 1. Let $I \subset k[V^*]$ be the ideal of $T$, $I_\mu = I \cap S^2_\mu(V)^*$, and, for $r = 6$, let $I_0 = I \cap S^3_0(V)^*$.

Let $\tilde{\mu}$ be the character by which $T$ acts on $S^2_\mu(V)$. The $T$-invariant hypersurface in $T$ cut by the zeros of a form from $S^2_\mu(V)^* \setminus I_\mu$ is mapped by $f : T \to X$ to a conic on $X$. The class of this conic in Pic$X$, up to sign, is $\tilde{\mu} \in \hat{T}$ under the isomorphism $\tilde{T} \simeq \text{Pic} X$ given by the type of the torsor $f : T \to X$ (see the comments before Prop. 6.2 in [13]). All conics on $X$ in a given class are obtained in this way; they form a 2-dimensional linear system, hence the codimension of $I_\mu$ in $S^2_\mu(V)^*$ is 2. Let $I^\perp_\mu \subset S^2_\mu(V)$ be the 2-dimensional zero set of $I_\mu$. The corresponding projective system defines a morphism $f_\mu : X \to \mathbb{P}^1 = \mathbb{P}(I^\perp_\mu)$ whose fibres are the conics of the class $\tilde{\mu}$. The link between $\text{Ver}_\mu$ and $f_\mu$ is described in the following commutative diagram:

\[ \begin{array}{ccc}
V & \supset (G/P)_a & \supset T & \to X \\
\text{Ver}_\mu & \downarrow & \downarrow f_\mu & \\
S^2_\mu(V) & \supset & p^\perp_\mu & \supset I^\perp_\mu \setminus \{0\} & \to \mathbb{P}^1
\end{array} \] (3)

Here $p^\perp_\mu$ is the zero set of $p_\mu \in S^2_\mu(V)^*$.

**Lemma 1.3** For $\mu \in W\omega_1$ the vertical maps in (3) are surjective, and $\dim S^2_\mu(V) = r - 1$.

**Proof** For the two right hand maps the statement is clear. The map $V \to S^2_\mu(V)$ is surjective because all eigenspaces of $T$ in $V$ are 1-dimensional. Since $\dim S^2_\mu(V)$
does not change if we replace \( \mu \) by \( w\mu \) for any \( w \in W \), to calculate \( \dim S^2_\mu(V) \) we can assume that \( \mu = \omega_1 \). But \( \omega_1 \) is a weight of \( H' \) in \( V_2 \), so we have \( S^2_{\omega_1}(V) = S^2_{\omega_1}(V_1) \oplus (V_2)_{\omega_1} \), where \( \dim (V_2)_{\omega_1} = 1 \). Starting with the case of the Plücker coordinates for \( r = 4 \), one shows by induction that \( \dim S^2_{\omega_1}(V) = r - 1 \). Hence \( \dim S^2_\mu(V) = r - 1 \) and so \( \dim p_\mu^\perp = r - 2 \) for any \( \mu \).

To compute \( \text{Ver}_\mu((G/P)_a) \) we can continue to assume that \( \mu = \omega_1 \). If \( x \in V_1 \), then \( \text{Ver}_{\omega_1} \) sends \( \exp(x) \in (G/P)_a \) to \( \text{Ver}_{\omega_1}(x) + p_{\omega_1}(x) \). Thus the projection of \( \text{Ver}_{\omega_1}((G/P)_a) \) to \( S^2_{\omega_1}(V_1) = \text{Ver}_{\omega_1}(V_1) \), which is a vector space of dimension \( r - 2 \), is surjective. Hence \( \text{Ver}_{\omega_1} \) maps \( (G/P)_a \) surjectively onto \( p_{\omega_1}^\perp \). QED

Arguing by induction as in this proof, it is easy to show starting with the case of the Plücker coordinates on \( \text{Gr}(2,5) \), that \( p_\mu(x) \) is the sum of all the monomials of weight \( \mu \) with non-zero coefficients.

## 2 Torsors over split del Pezzo surfaces

Unless stated otherwise we assume that \( r \geq 5 \), so that \( G' \) is of type \( A_4, D_5 \) or \( E_6 \). Recall that we use dashes to denote objects related to the ‘previous’ root system.

Let \( x_0 \) be a \( k \)-point of \( T'^\times \). We define the dense open subset \( \Omega(x_0) \subset (G'/P')_a^\times \) as the set of \( k \)-points \( x \) such that \( \exp(x_0^{-1} x T'^\times) \) is not contained in \( V \setminus V^\times \), that is, in the union of weight hyperplanes of \( V \). For \( r = 5 \) or \( 6 \) the set \( \Omega(x_0) \) is the complement to the union of the closed subsets

\[
Z_\mu(x_0) = \{ x \in (G'/P')_a \mid p_\mu(x_0^{-1}xu) \in I'_\mu \}
\]

for all weights \( \mu \) of \( V_2 \); for \( r = 7 \) one also removes the closed subset

\[
Z_0(x_0) = \{ x \in (G'/P')_a \mid q(x_0^{-1}xu) \in I'_0 \}.
\]

The condition \( y_0 \in \Omega(x_0) \) implies that for all \( \mu \) the vectors \( \text{Ver}_\mu(x_0) \) and \( \text{Ver}_\mu(y_0) \) are not proportional. Since \( \dim S^2_\mu(V_1)^* = 2 + \dim I'_\mu \), we see that for any \( y_0 \in T' \cap \Omega(x_0) \) the subspace \( I'_\mu \subset S^2_\mu(V_1)^* \) consists of the forms vanishing at \( x_0 \) and \( y_0 \). The ideal \( I' \) is generated by the \( I'_\mu \), so \( T' \) is uniquely determined by any two of its points satisfying a certain open condition.

Recall that \( \pi : V \to V_1 \) is the natural projection, cf. (2).

**Lemma 2.1** Let \( x_0 \) be a \( k \)-point of \( T'^\times \), let \( y_0 \) be a \( k \)-point of \( \Omega(x_0) \cap T' \), and let \( T \subset (G/P)_a^\times \) be the torsor defined by the triple \( (T', x_0, y_0) \) as described above. Then we have the following statements.

(i) The closed set \( Z_\mu(x_0) \subset (G'/P')_a \) consists of the points \( x \in (G'/P')_a(\overline{k}) \) such that \( p_\mu(x^{-1}y_0x) = 0 \). For \( r = 7 \) the closed set \( Z_0(x_0) \) consists of the points \( x \in (G'/P')_a(\overline{k}) \) such that \( q(x^{-1}y_0x) = 0 \).
(ii) The open set \( \Omega(x_0) \cap T' \) is the inverse image of the complement to all exceptional curves on \( X' \) and to all conics on \( X' \) passing through \( f'(x_0) \). For \( r = 7 \) one also removes from the cubic surface \( X' \subset \mathbb{P}^3 \) the nodal curve cut by the tangent plane to \( X' \) at \( f'(x_0) \). We have \( T^x = \pi^{-1}(\Omega(x_0) \cap T') \).

(iii) We have \( t = \exp(x_0^{-1}y_0^3) \in T^x \).

Proof (i) The inclusion of \( Z_\mu(x_0) \) into the hypersurface given by \( p_\mu(x_0^{-1}y_0x) = 0 \) is clear: assigning the variable \( u \) the value \( y_0 \in T' \) we see that \( p_\mu(x_0^{-1}xu) \in I_\mu' \) implies that \( p_\mu(x_0^{-1}xy_0) = 0 \). Conversely, let us prove that every point \( x \) of \( (G'/P')_a \) satisfying the condition \( p_\mu(x_0^{-1}y_0x) = 0 \), is in \( Z_\mu(x_0) \). Using Lemma 1.3 we see that the set of quadratic forms \( p_\mu(x_0^{-1}yu) \) on \( V_1 \) for a fixed \( x_0 \) and arbitrary \( y \in (G'/P')_a \) is a vector subspace \( L \subset S^2_\mu(V_1)^* \) of codimension 1, in fact this is the space of forms vanishing at \( x_0 \). As was pointed out before the statement of the lemma, \( I_\mu' \) is the subspace of \( L \) of codimension 1 consisting of the forms vanishing at \( y_0 \). This proves the desired inclusion.

Now let \( r = 7 \). The inclusion of \( Z_0(x_0) \) into the hypersurface \( q(x_0^{-1}y_0x) = 0 \) is clear for the same reason as above. Conversely, let \( x \in (G'/P')_a \) be such that \( q(x_0^{-1}xu) = 0 \). We need to prove that \( q(x_0^{-1}xu) \) vanishes for any \( \bar{k} \)-point \( u \) of \( T' \). In the end of the proof of Prop. 6.2 of [13] we showed that the dual space \( H^0(X', \mathcal{O}(-K_{X'}))^* \) is a 4-dimensional vector subspace of \( S^3_0(V_1) \), so that we have a commutative diagram similar to (3):

\[
\begin{array}{ccc}
V_1 & \subset & T' \\
\vee_{\mathcal{O}} & \downarrow & \uparrow \varphi \\
S^3_0(V_1) & \subset & H^0(X', \mathcal{O}(-K_{X'}))^* \setminus \{0\} \to \mathbb{P}(H^0(X', \mathcal{O}(-K_{X'}))^*)
\end{array}
\]

where \( \varphi \) is the anticanonical embedding \( X' \hookrightarrow \mathbb{P}^3 \). In loc. cit. we also showed that for any \( x \in (G'/P')_a(\bar{k}) \) the cubic form \( q(x_0^{-1}xu) \), considered as a linear form on \( S^3_0(V_1) \), vanishes on the tangent space \( T_{x_0} \cong \mathbb{P}^2 \) to \( \varphi(X') \subset \mathbb{P}^3 \) at \( \varphi f'(x_0) \). It is thus obvious that if \( q(x_0^{-1}xu) \) vanishes at any point of \( \varphi(X') \) outside of \( T_{x_0} \), then \( q(x_0^{-1}xu) \) vanishes at any \( \bar{k} \)-point \( u \) of \( T' \). But \( \varphi f'(y_0) \notin T_{x_0} \), otherwise \( q(x_0^{-1}y_0) = 0 \) for any \( \bar{k} \)-point \( z \) of \( (G'/P')_a \) contradicting the assumption that \( y_0 \) is in \( \Omega(x_0) \). Thus \( q(x_0^{-1}y_0) = 0 \) implies that \( q(x_0^{-1}xu) \in I_0' \).

(ii) The geometric description of \( \Omega(x_0) \cap T' \) follows from [13], Cor. 6.3. Hence \( \Omega(x_0) \cap T' \) is obtained from \( T' \) by removing the images \( \pi(E) \) of all exceptional divisors \( E \subset T \), so that \( \pi(T^x) = \Omega(x_0) \cap T' \).

(iii) Recall that \( \exp(x) \) gives a section of the natural morphism \( \pi: T \to x_0^{-1}y_0 T' \) over the complement to the fibre \( T'_{y_0} \). Thus \( t \in T \). Since \( y_0 \) is in \( \Omega(x_0) \cap T' \) we see from (ii) that \( t \) is in \( T^x \). QED

Let \( T \subset V^a \) be a closed \( T \)-invariant subvariety such that \( T/T \) is a split del Pezzo surface and the weight hyperplane sections of \( T \) are exceptional divisors with
multiplicity 1. The torsor $\mathcal{T}$ defines an important subset of the torus $S$. Namely, let $Z$ be the closed subset of $S$ consisting of the points $s$ such that $s\mathcal{T} \subset (G/P)_a$. Equivalently, $Z = \bigcap_{x \in T^x} x^{-1}(G/P)\ast$. The set $Z$ is $T$-invariant, since such are $(G/P)_a$ and $\mathcal{T}$. In the case when $\mathcal{T} \subset (G/P)_a$, the variety $Z$ contains the identity element $1 \in S(k)$.

**Lemma 2.2** Under the assumptions of Lemma 2.1 for $r = 4$ we have $Z = T$, and for $r \geq 5$ we have $\pi(Z) = y_0^{-1}\Omega(x_0)$ which is dense and open in $y_0^{-1}(G'/P')\ast$. The closed subvariety $Z \subset S$ is the affine cone (without zero) over $s^{-1}\exp(x_0^{-1}y_0\Omega(x_0))$; in particular, $Z$ is geometrically integral, and $t^{-1}T^x \subset Z$. For $r = 5$ this inclusion is an equality.

**Proof** The statement in the case $r = 4$ is clear since $\mathcal{T}$ is dense in $(G/P)_a$, and the only elements of $S$ that leave $G/P = \text{Gr}(2,5)$ invariant are the elements of $T$. (Indeed, it is well known that the group of relations among the classes of 10 exceptional curves on $X$ is generated by the quadratic relations given by degenerate elements of conic pencils on $X$. These quadratic relations are in a natural bijection with the quadratic equations among the Plücker coordinates of $\text{Gr}(2,5)$.) Now assume that $r \geq 5$. For a fixed $x_0$, in order to construct an embedding $\mathcal{T} \subset (G/P)_a$ we can choose any $y$ in the dense open subset $\Omega(x_0) \subset (G'/P')_a$. The embeddings defined by $(x_0, y_0)$ and $(x_0, y)$ satisfy the conditions of Lemma 1.1. We obtain an element $s \in Z$ such that $\pi(s) = y_0^{-1}y$. Thus $\pi(Z)$ contains $y_0^{-1}\Omega(x_0)$.

Let us prove that $\pi(Z) \subset y_0^{-1}(G'/P')\ast$. Let $\pi_0 : V \to V_0 \simeq k$ be the natural projection. Choose $y \in \mathcal{T} \subset (G/P)_a$ such that $\pi(y) = y_0 \in \Omega(x_0) \subset (G'/P')\ast$. By Lemma 4.1 of [13] we have $\pi_0(y) = 0$. Thus $\pi_0(sy) = 0$ for any $s \in Z$. But since $sy \in (G/P)_a$, an inspection of cases in Lemma 4.1 of [13] shows that $\pi(sy) = \pi(s)y_0 \in (G'/P')\ast$. Therefore, $\pi(Z) \subset y_0^{-1}(G'/P')\ast$. Next, we note that $st \in (G/P)_a$ (since $t \in T^x$ by Lemma 2.1). The coordinates of the projection of $st$ to $V_2$ equal $p_\mu(\pi(s)x_0^{-1}y_0^2)$, up to a non-zero constant, hence $p_\mu(\pi(s)x_0^{-1}y_0^2) \neq 0$ for all $\mu$. But for $r \leq 6$ the open set $y_0^{-1}\Omega(x_0) \subset y_0^{-1}(G'/P')\ast$ is given by $p_\mu(x_0^{-1}y_0^2u) \neq 0$, by Lemma 2.1 (i). For $r = 7$ a similar argument shows that $q(\pi(s)x_0^{-1}y_0^3) \neq 0$. Thus we obtain the equality $\pi(Z) = y_0^{-1}\Omega(x_0)$.

By Lemma 2.1 (iii), $t = \exp(x_0^{-1}y_0^2)$ is in $T^x$ so we have $tZ \subset (G/P)_a$. Since $Z$ is invariant under the action of $G_m = \{y_t | t \in \mathbb{F}^\times\}$, we see from Lemma 4.1 of [13] that $Z$ is a $G_m$-torsor over $\pi(Z) = y_0^{-1}\Omega(x_0)$. Moreover, $t^{-1}\exp(x_0^{-1}y_0^2x)$ is a section of this torsor. This proves that $Z$ is the affine cone over $t^{-1}\exp(x_0^{-1}y_0\Omega(x_0))$.

If $r = 5$, then $\Omega(x_0)$ is a dense open subset of $T'$ as both sets are Zariski open in $\text{Gr}(2,5)$. Thus the last statement follows from Lemma 2.1 (ii). QED

**Definition 2.3** $r - 3$ points $z_0, \ldots, z_{r-4}$ in $Z(\mathbb{K})$ are in general position if for any weight $\mu \in W_\omega_1$ the vectors $\text{Ver}_\mu(z_i)$, $i = 0, \ldots, r - 4$, are linearly independent.
More precisely, for any $k$

Proof

By Corollary 1.2 it is enough to prove the theorem for $T_{G/P}$. The torsor $T$ result gives a concise natural description of these equations, in terms of the well assumptions of Lemma 2.1. The torsor $T_{G/P}$ is Zariski dense subset of a vector space of dimension $r - 3$, by Lemma 1.3, we see that $V_{G/P}$ is in the complement to the union of the inverse images under $V_{G/P}$ of Lemma 1.3, we have $S^2_1(V) = S^2_1(V_1) \oplus (V_2)_1$. The image of $tZ$ consists of the points $V_{\omega_1}(x_0 y_0 u) + p_{\omega_1}(x_0 y_0 u)$, where $u$ is in $\Omega(x_0)$, by Lemma 2.2. Since $V_{\omega_1}$ sends $(G'/P)'_a$ to a vector space of dimension $r - 3$, by Lemma 1.3, we see that $V_{\omega_1}(tZ)$ is a dense subset of a vector space of this dimension. Hence the same is true for $Z$.

We can choose the points $z_1, \ldots, z_{r-4}$ in $Z(k)$ one by one, in such a way that $z_n$ is in the complement to the union of the inverse images under $V_{\mu}$ of the linear span of $V_{\mu}(z_i)$, $i = 0, \ldots, n - 1$. This complement is non-empty since $V_{\mu}(Z)$ is a Zariski dense subset of a vector space of dimension $r - 3$. QED

Equations for $T$ have been given by Popov [11] and Derenthal [7]. The following result gives a concise natural description of these equations, in terms of the well known equations of $(G/P)_a \subset V$.

Theorem 2.5 Let $r = 4, 5, 6$ or 7. Every split del Pezzo surface $X$ of degree $9 - r$ has a universal torsor $T$ which is an open subset of the intersection of $r - 3$ dilatations of $(G/P)_a$ by $k$-points of the diagonal torus $S$. In the above notation we have

$$T^\times = \bigcap_{z \in \mathcal{Z}(\mathbb{F})} z^{-1}(G/P)_a^\times = \bigcap_{i=0}^{r-4} z_i^{-1}(G/P)_a^\times,$$

where $z_0 = 1, z_1, \ldots, z_{r-4}$ are $k$-points of $Z$ in general position.

Proof By Corollary 1.2 it is enough to prove the theorem for $T$ which satisfies the assumptions of Lemma 2.1. The torsor $T$ is clearly contained in the closed set $S = \cap_{s \in \mathcal{Z}(\mathbb{F})} s^{-1}(G/P)_a \subset V$. Since $T^\times$ is closed in $V^\times$, the density of $T$ in $S$ implies $T^\times = S^\times$. To prove this density it is enough to show that $x_0^{-1} y_0 T'$ is dense in $\pi(S)$.

For $v \in V \otimes \mathbb{F}$ we write $v = (v_0, v_1, v_2, v_3)$, where $v_i \in V_i \otimes \mathbb{F}$. Similarly, we write $s \in S(\mathbb{F})$ as $(s_0, s_1, s_2, s_3)$, where $s_i \in \text{GL}(V_i \otimes \mathbb{F})$. In this notation the set

$$\bigcap_{s \in \mathcal{Z}(\mathbb{F})} \{(s_0^{-1} t, s_1^{-1} t x, s_2^{-1} t p(x), s_3^{-1} t q(x)) | x \in V_i \otimes \mathbb{F}, t \in \mathbb{F}^\times\}$$

is dense in $S$. This set can also be written as

$$\bigcap_{s \in \mathcal{Z}(\mathbb{F})} \{(t, x, (ts_0)^{-1} s_2^{-1} p(s_1 x), (ts_0)^{-2} s_3^{-1} q(s_1 x)) | x \in V_i \otimes \mathbb{F}, t \in \mathbb{F}^\times\}. $$
Since $(1, 1, 1, 1) \in \mathcal{Z}$, we see that $\pi(\mathcal{S})$ is contained in the set of $x \in V_1 \otimes \mathbb{k}$ such that for all $s \in \mathcal{Z}(\mathbb{k})$ we have

$$s_0^{-1}s_2^{-1}p(s_1x) = p(x).$$

Let $J \subset k[V_1^*]$ be the ideal of $x_0^{-1}y_0T'$, and $J_\mu = J \cap S^2_{\mu}(V_1)^*$. In the same way as $I_\mu$, the ideal $J_\mu$ has codimension 2 in $S^2_{\mu}(V_1)^*$. Lemma 1.3 implies that the linear span $L$ of the quadratic forms $p_\mu(y_0x_0^{-1}x)$ on $V_1$ for a fixed $y_0 \in (G'/P')^*_a$ and arbitrary $y \in (G'/P')^*_a$ has codimension 1 in $S^2_{\mu}(V_1)^*$ (in fact, $L$ is the space of forms vanishing at $y_0$). Lemma 2.2 implies that $L$ coincides with the linear span of the quadratic forms $p_\mu(s_1x)$, for all $s \in \mathcal{Z}$. Hence the linear span of the forms $s_0^{-1}s_2^{-1}p_\mu(s_1x) - p_\mu(x)$, for all $s \in \mathcal{Z}$, has codimension at most 2 in $S^2_{\mu}(V_1)^*$. However, the inclusion $x_0^{-1}y_0T' \subset \pi(\mathcal{S})$ implies that this space is in $J_\mu$, and thus coincides with $J_\mu$. This holds for every $\mu$, and the ideal $J$ is generated by the $J_\mu$ (since the same is true for $I_\mu$), therefore $x_0^{-1}y_0T'$ is dense in $\pi(\mathcal{S})$.

Let us prove the second equality in (4). It is well known that the intersection of the ideal of $(G/P)_a$ with $S^2_{\mu}(V^*)$ is 1-dimensional; let $P_\mu(u)$ be a non-zero element in this intersection. Then $P_\mu(z_iu)$, where $z_0, \ldots, z_{r-4}$ are in general position, span a vector space of dimension $r - 3$ contained in the intersection of the ideal of $T$ with $S^2_{\mu}(V^*)$, which has the same dimension. Thus $P_\mu(z_iu)$, $i = 0, \ldots, r - 4$, is a complete system of equations of $T$ of weight $\mu$. This completes the proof. QED

Remark In the case $r = 5$ the general position condition has a clear geometric interpretation. By the last claim of Lemma 2.2 we have $T^\times = s\mathcal{Z}$ for some $s \in S(k)$ well defined up to $T(k)$. If $T \subset (G/P)_a$, then $\mathcal{Z}$ contains 1, to that $s$ is a $k$-point of $T^\times$. Then the previous theorem implies

$$T^\times = s\mathcal{Z} = (G/P)_a^\times \cap r^{-1}s(G/P)_a^\times,$$

where $r$ is a $k$-point in $T^\times$ such that $f(s)$ and $f(r)$ are points in $X^\times$ not contained in a conic on $X$, cf. diagram (3). Here $f(s)$ is uniquely determined by $T$, whereas $r$ can be any point in the open subset of $X$ given by this condition.

This remark can be seen as a particular case of the following description of $\mathcal{Z}$. For any $g$ and $h$ in $T^\times(k)$ such that $Ver_\mu(h)$ and $Ver_\mu(g)$ are not proportional for any $\mu \in W\omega_1$, we have $\mathcal{Z} = g^{-1}(G/P)_a^\times \cap h^{-1}(G/P)_a^\times$. The proof is similar to that of Theorem 2.5; we omit it here since we shall not need this fact.

To construct $r - 3$ points in $\mathcal{Z}$ in general position is not hard, because the points of $\mathcal{Z}$ are parameterized by polynomials. Indeed, decompose $V_1 = V_{1,0} \oplus V_{1,1} \oplus V_{1,2}$ similarly to (2), and consider the points $t^{-1}\exp(x_0^{-1}y_0\exp(v_i))$, where $v_1, \ldots, v_{r-3}$ in $V_{1,1}$ satisfy certain open conditions which are easy to write down using Lemma 2.2.
3 A uniqueness result

The choice of $y_0$ plays the role of a ‘normalization’ for the embedding of a torsor into $(G/P)_a$. It is convenient to choose these normalizations in a coherent way. Let $M_1, \ldots, M_r$ be $k$-points in general position in $\mathbb{P}^2$, and let $X_r$ be the blowing-up of $\mathbb{P}^2$ in $M_1, \ldots, M_r$. The complement to the union of exceptional curves $X_r^\times \subset X_r$ can be identified with an open subset $U \subset \mathbb{P}^2$. Choose $u_0 \in U(k)$. At every step of our inductive process we can choose the points $y_0$ in the fibre of $\mathcal{T}' \to X'$ over $u_0$. Thus we get a compatible family of the $y_0$ (more precisely, of torus orbits) that are mapped to each other by the surjective maps $\mathcal{T} \to \mathcal{T}'$. In our previous notation, the point $t = \exp(x_0^{-1}y_0^2)$ must be taken for the point $y_0$ of the next step.

If $A$ is a subset of the torus $S$, then we denote by $P^n(A) \subset S$ the set of products of $n$ elements of $A$ in $S$. We define $P^0(A) = T$.

**Proposition 3.1** Let $r$ and $n$ be integers satisfying $4 \leq r \leq 7$, $0 \leq n \leq r - 4$. Under the assumptions of Lemma 2.1, if at every step of our construction we choose the points $y_0$ over a fixed point of $U$, then we have the following statements:

(i) $P^{n+1}(t^{-1}\mathcal{T}^\times) \subset t^{-1}(G/P)_a^\times$,
(ii) $P^n(t^{-1}\mathcal{T}^\times) \subset Z$.

**Proof** (i) and (ii) are clearly equivalent. For $n = 0$ the inclusion (i) is the main theorem of [13], and this also covers the case $r = 4$. Let $n \geq 1$. Recall that the projection $\pi$ maps $t^{-1}\mathcal{T}$ onto $y_0^{-1}\mathcal{T}'$. Assume that we have the desired inclusions for $n - 1$ and for both torsors $\mathcal{T}'$ and $\mathcal{T}$, namely

$$P^n(t^{-1}\mathcal{T}^\times) \subset t^{-1}(G/P)_a^\times, \quad P^n(y_0^{-1}\mathcal{T}'^\times) \subset y_0^{-1}(G/P)_a^\times.$$  

By Lemma 4.1 of [13] every $\overline{k}$-point of $(G/P)_a^\times$ can be written as $g_x \cdot \exp(v)$, where $x \in \overline{k}$, and $v \in V_1 \otimes \overline{k}$. By the first inclusion in induction assumption this is also true for elements of $tP^n(t^{-1}\mathcal{T}^\times)$. Since $\mathcal{T}$ is $g_x$-invariant, we have $\exp(v) \in tP^n(t^{-1}\mathcal{T}^\times)$. On applying $\pi$ to both sides we deduce $v \in x_0^{-1}y_0^2P^n(y_0^{-1}\mathcal{T}'^\times)$. Applying the second inclusion in induction assumption we obtain $v \in x_0^{-1}y_0(G'/P')_a^\times$. Therefore, $P^n(t^{-1}\mathcal{T}^\times)$ is contained in the affine cone over $t^{-1}\exp(x_0^{-1}y_0(G'/P')_a^\times)$. But this implies $P^n(t^{-1}\mathcal{T}^\times) \subset Z$, since $tZ$ is the intersection of the affine cone over $\exp(x_0^{-1}y_0(G'/P')_a^\times)$ with $V^\times$, by the last statement of Lemma 2.2. This proves (ii), and hence also (i). QED

**Proposition 3.2** Let $r = 4, 5, 6$ or $7$, and let $\mathcal{T} \subset V^{sf}$ be a closed $T$-invariant subvariety such that $X = \mathcal{T}/T$ is a split del Pezzo surface of degree $9 - r$, and the weight hyperplane sections of $\mathcal{T}$ are exceptional divisors with multiplicity 1. Let $Z \subset S$ be the closed subset of points $z$ such that $z\mathcal{T} \subset (G/P)_a$. Then there is a unique $s \in S(k)$ defined up to an element of $T(k)$, such that $P^{r-4}(\mathcal{T}^\times) \subset sZ$.  

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Proof By Corollary 1.2, up to translating $T$ by an element of $S(k)$, we can assume that $T \subset (G/P)_a$ is obtained by our construction. Thus the existence of $s$ follows from Proposition 3.1. We prove the uniqueness by induction in $r$. For $r = 4$ the statement is clear, since the only elements of $S$ that leave $Gr(2,5)$ invariant are the elements of $T$ (see the proof of Lemma 2.2).

Assume $r \geq 5$. By Lemma 2.2, $P^{r-4}(t^{-1}T^\times) \subset sZ$ implies $P^{r-4}(y_0^{-1}T^\times) \subset \pi(s)\gamma^{-1}(G'/P')_a$, from which it follows that $P^{r-5}(y_0^{-1}T^\times) \subset \pi(s)Z'$. By induction assumption $\pi(s)$ is unique up to an element of $T'(k)$. Therefore, $s$ is unique up to an element of $T(k)$. QED

Remark For $r = 5$ the inclusion $P^{r-4}(T^\times) \subset sZ$ is an equality by the last claim of Lemma 2.2, but this is no longer so for $r = 6$ or $7$, for dimension reasons.

4 Non-split del Pezzo surfaces

Let $\Gamma = Gal(\overline{k}/k)$. Let $G$ be a split simply connected semisimple group over $k$ with a split maximal $k$-torus $H$ and the root system $R$. There is a natural exact sequence of algebraic $k$-groups

$$1 \to H \to N \to W \to 1,$$

(6)

where $N$ is the normalizer of $H$ in $G$, and $W$ is the Weyl group of $R$. The action of $N$ by conjugation gives rise to an action of $W$ on the torus $H$. Since $H$ is split, the Galois group $\Gamma$ acts trivially on $W$. Thus the continuous 1-cocycles of $\Gamma$ with values in $W$ are homomorphisms $\Gamma \to W$, and the elements of $H^1(k,W)$ are homomorphisms $\Gamma \to W$ considered up to conjugation in $W$.

Theorem 4.1 (Gille–Raghunathan) For any $\sigma \in \text{Hom}(\Gamma,W)$ the twisted torus $H_\sigma$ is isomorphic to a maximal torus of $G$.

Proof See [8], Thm. 5.1 (b), or [12], Thm. 1.1. QED

Recall from [14], I.5.4, that (6) gives rise to the exact sequence of pointed sets

$$1 \to N(k) \to G(k) \to (G/N)(k) \xrightarrow{\varphi} H^1(k,N) \to H^1(k,G).$$

(Note by the way that the last map here is surjective.) The homogeneous space $G/N$ is the variety of maximal tori of $G$, so that an equivalent form of the Gille–Raghunathan theorem is the surjectivity of the composite map

$$(G/N)(k) \to H^1(k,N) \to H^1(k,W) = \text{Hom}(\Gamma,W)/\text{Inn}(W),$$

where $\text{Inn}(W)$ is the group of inner automorphisms of $W$. We fix an embedding of $H_\sigma$ as a maximal torus of $G$, this produces a $k$-point $[H_\sigma]$ in $G/N$. The choice of a $\overline{k}$-point $g_0$ in $G$ such that $g_0Hg_0^{-1} = H_\sigma$ defines a 1-cocycle $\rho : \Gamma \to N(\overline{k}),$
\( \rho(\gamma) = g_0^{-1} \cdot \gamma g_0 \), which is a lifting of \( \sigma \in Z^1(k,W) = \text{Hom}(\Gamma,W) \). We have \([\rho] = \varphi[H_\sigma] \) (\cite{ibidem}, ibidem), moreover, the image of \([\rho] \) in \( H^1(k,G) \) is trivial.

Let \( G \to \text{GL}(V) \) be an irreducible representation of \( G \). Define \( T \subset \text{GL}(V) \) as a torus generated by \( H \) and the scalar matrices \( G_m \). The group \( N \) acts by conjugation on \( T \). The twisted torus \( T_\sigma \) is just the extension of \( H_\sigma \) by scalar matrices.

Let \( (G/P)_a \subset V \) be the orbit of the highest weight vector (with zero added to it); \( P \subset G \) is a parabolic subgroup, and \( (G/P)_a \) is the affine cone over \( G/P \). The maximal torus \( H_\sigma \subset G \) acts on \( (G/P)_a \), and so does \( T_\sigma \). Define \( U_\sigma \) to be the dense open subset of \( (G/P)_a \) consisting of the points with closed \( H_\sigma \)-orbits and trivial stabilizers in \( T_\sigma \).

The group \( N \subset G \) acts on \( V \) preserving \( V^{\text{sf}} \) and \( V^\times \), thus giving rise to the action of \( W \) on \( V^{\text{sf}}/T \) and on \( V^\times/T \) by automorphisms of algebraic varieties (not necessarily preserving some group structure on \( V^\times/T \)). The action of \( N \) preserves \( (G/P)^{\text{sf}}_a \subset V \), thus \( W \) acts on \( Y = (G/P)^{\text{sf}}_a/T \). Hence we define the twisted forms \( V^{(G/P)\sigma}_a, (V^\times/T)_\sigma \) and \( Y_\sigma \). The variety \( (V^\times/T)_\sigma \) is an open subset of the quasi-projective toric variety \( (V^{\text{sf}}/T)_\sigma \), which contains \( Y_\sigma \) as a closed subset.

**Lemma 4.2** The \( k \)-varieties \( Y_\sigma \) and \( U_\sigma/T_\sigma \) are isomorphic.

**Proof** Recall that \( g_0 \in G(\overline{k}) \) is a point such that \( \rho(\gamma) = g_0^{-1} \cdot \gamma g_0 \in Z^1(k,N) \) is a cocycle that lifts \( \sigma \in Z^1(k,W) = \text{Hom}(\Gamma,W) \). The group \( N \) acts on the homogeneous space \( (G/P)_a \) as a subgroup of \( G \), so we can define \( (G/P)_{a,\rho} \) as the twist of \( (G/P)_a \) by \( \rho \). It is immediate to check that the map \( x \mapsto g_0 x \) on \( \overline{k} \)-points of \( (G/P)_a \) gives rise to an isomorphism of \( k \)-varieties \( (G/P)_{a,\rho} \rightarrow (G/P)_a \). If \( G_\rho \) is the inner form of \( G \) defined by \( \rho \), then \( G_\rho \) acts on \( (G/P)_{a,\rho} \) on the left. The embedding \( H \hookrightarrow G \) gives rise to an embedding \( H_\sigma \hookrightarrow G_\rho \), so that \( T_\sigma \) acts on \( (G/P)_{a,\rho} \) on the left. On the other hand, \( T_\sigma \) also acts on \( (G/P)_a \) on the left. It is straightforward to check that the isomorphism \( (G/P)_{a,\rho} \rightarrow (G/P)_a \) is \( T_\sigma \)-equivariant.

Let \( (G/P)^{\text{sf}}_{a,\rho} \) be the subset of \( (G/P)_{a,\rho} \) consisting of the points with closed \( H_\sigma \)-orbits with trivial stabilizers in \( T_\sigma \). The closedness of orbits and the triviality of stabilizers are conditions on \( \overline{k} \)-points, hence we obtain a \( T_\sigma \)-equivariant \( k \)-isomorphism \( (G/P)^{\text{sf}}_{a,\rho} \rightarrow U_\sigma \). It descends to an isomorphism \( Y_\sigma \rightarrow U_\sigma/T_\sigma \). This proves the lemma. QED

**Corollary 4.3** For any homomorphism \( \sigma : \Gamma \rightarrow W \) the twisted variety \( Y_\sigma \) has a \( k \)-point, and so does \( (V^\times/T)_\sigma \).

**Proof** Since \( k \) is an infinite field, any dense open subset of \( (G/P)_a \) contains \( k \)-points. Thus \( Y_\sigma^\times(k) \neq \emptyset \), but this is a subset of \( (V^\times/T)_\sigma \), so that this variety has a \( k \)-point. QED

**Remark** This approach via the Gille–Raghunathan theorem generalizes a key ingredient in the second author’s proof of the Enriques–Swinnerton-Dyer theorem that
every del Pezzo surface of degree 5 has a $k$-point, from quotients of Grassmannians by the action of a maximal torus to quotients of arbitrary homogeneous spaces of quasi-split semisimple groups. We plan to return to this more general statement in another publication.

We now assume that $R$ is the root systems of rank $r$ in (1), and that the highest weight of the $G$-module $V$ is the fundamental weight dual to the root indicated in (1). Then $V$ is minuscule, so that the centralizer $S$ of $H$ in $\text{GL}(V)$ is a torus. Let $R = S/T$. We obtain an exact sequence of $k$-tori:

$$1 \to T \to S \to R \to 1. \quad (7)$$

The group $N$ acts by conjugation on $T$ and hence also on $S$ and $R$. The connected component of 1 acts trivially, so we obtain an action of $W$ on these tori (preserving the group structure). On twisting $T$, $S$ and $R$ by $\sigma$ we obtain an exact sequence of $k$-tori:

$$0 \to T_\sigma \to S_\sigma \to R_\sigma \to 0. \quad (8)$$

Note in passing that the character group $\hat{S}$ has an obvious $W$-invariant basis, which gives rise to a Galois invariant basis of $\hat{S}_\sigma$. In other words, $S_\sigma$ is a quasi-trivial torus; in particular, $H^1(k, S_\sigma) = 0$ as follows from Hilbert’s theorem 90. Note also that $V^\times/T$ is a torsor under $R$, so that $(V^\times/T)_\sigma$ is a torsor under $R_\sigma$. By Corollary 4.3 this torsor is trivial, that is, there is a (non-canonical) isomorphism $(V^\times/T)_\sigma \simeq R_\sigma$.

Let $X$ be a del Pezzo surface over $k$, not necessarily split, of degree $9-r$, where $r$ is the rank of the root system $R$. Let $\overline{X}$ be the surface obtained from $X$ by extending the ground field from $k$ to $\overline{k}$. We write $X^\times$ for the complement to the union of exceptional curves on $X$. Our construction identifies $\hat{S}$ with the free abelian group $\text{Div}_{X\setminus\overline{X}}\overline{X}$ generated by the exceptional curves on $\overline{X}$, and $\hat{T}$ with $\text{Pic}\overline{X}$ (via the type of the universal torsor $T \to X$). The Galois group permutes the exceptional curves on $\overline{X}$, thus defining a homomorphism $\sigma_X : \Gamma \to W$, where $W$ is the Weyl group of $R$. This homomorphism is well defined up to conjugation in $W$, so we have a well defined class $[\sigma_X] \in H^1(\Gamma, W)$, where $\Gamma$ acts trivially on $W$.

We now assume $\sigma = \sigma_X$. Then we get isomorphisms of $\Gamma$-modules

$$\hat{S}_\sigma = \text{Div}_{\overline{X}\setminus\overline{X}^\times}\overline{X}, \quad \hat{T}_\sigma = \text{Pic}\overline{X},$$

thus $T_\sigma$ is the Néron–Severi torus of $X$. The kernel of the obvious surjective map $\text{Div}_{\overline{X}\setminus\overline{X}^\times}\overline{X} \to \text{Pic}\overline{X}$ is $\overline{k}[X^\times]^*/\overline{k}^*$, hence the dual sequence of (8) coincides with the natural exact sequence of $\Gamma$-modules

$$0 \to \overline{k}[X^\times]^*/\overline{k}^* \to \text{Div}_{\overline{X}\setminus\overline{X}^\times}\overline{X} \to \text{Pic}\overline{X} \to 0. \quad (9)$$

There is a natural bijection between the morphisms $X^\times \to R_\sigma$ and the homomorphisms of $\Gamma$-modules $\hat{R}_\sigma \to \overline{k}[X^\times]^*$. Universal torsors on $X$ exist if and only if the
exact sequence of $\Gamma$-modules

$$1 \to \bar{k}^* \to \bar{k}[X^*]^* \to \bar{k}[X^*]^*/\bar{k}^* \to 1$$

(10)

is split ([15], Cor. 2.3.10). Any splitting of this sequence gives a map

$$\tilde{R}_\sigma = \bar{k}[R_\sigma]^*/\bar{k}^* = \bar{k}[X^*]^*/\bar{k}^* \to \bar{k}[X^*]^*,$$

and hence defines a morphism $\phi : X^* \to R_\sigma$. By the ‘local description of torsors’ (see [4], 2.3 or [15], Thm. 4.3.1) the restriction of a universal $X$-torsor to $X^*$ is the pull-back of the torsor $S_\sigma \to R_\sigma$ to $X^*$ via $\phi$. Moreover, this gives a bijection between the splittings of (10) and the universal $X$-torsors. In our case it is easy to see that $\phi$ is an embedding. The isomorphism $\tilde{R}_\sigma = \bar{k}[X^*]^*/\bar{k}^*$ comes from our construction, thus after extending the ground field to $\bar{k}$, the morphism $\phi$ coincides, up to translation by a $\bar{k}$-point of $R$, with the embedding of $\bar{X}^*$ into $(V \otimes_k \bar{k})^*/\bar{T}$ obtained from the embedding $\overline{T}^* \subset \overline{V}^*$.

**Theorem 4.4** Let $r = 4, 5, 6$ or $7$. Let $X$ be a del Pezzo surface of degree $9 - r$ with a $k$-point, and let $\sigma \in H^1(\mathcal{I}, W)$ be the class defined by the action of the Galois group on the exceptional curves of $X$. There exists an embedding $X \hookrightarrow Y_\sigma$ such that the divisors in $Y_\sigma \setminus Y_\sigma^*$ cut the exceptional curves on $X$ with multiplicity $1$. The restriction of $U_\sigma \to Y_\sigma$ to $X \subset Y_\sigma$ is a universal $X$-torsor whose type is the isomorphism $\hat{T}_\sigma = \text{Pic} \bar{X}$.

**Proof** From Corollary 4.3 we get an embedding $Y_\sigma \hookrightarrow R_\sigma$, which becomes unique if we further assume that a given $k$-point of $Y_\sigma$ goes to the identity element of $R_\sigma$.

Since $X(k) \neq \emptyset$, there is a unique embedding $\phi : X^* \to R_\sigma$ such that the induced map $\phi^* : \tilde{R}_\sigma \to \bar{k}[X^*]^*$ is a lifting of the isomorphism $\tilde{R}_\sigma = \bar{k}[R_\sigma]^*/\bar{k}^* = \bar{k}[X^*]^*/\bar{k}^*$, and $\phi$ sends a given $k$-point of $X^*$ to 1.

Let $\mathcal{L}$ be the $k$-subvariety of the torus $R_\sigma$ whose points are $c \in R_\sigma(\bar{k})$ such that $c X^* \subset Y_\sigma^*$, where the multiplication is the group law of $R_\sigma$. To prove the first statement we need to show that $\mathcal{L}(k) \neq \emptyset$. Let $P^n(X^*)$ be the $k$-subvariety of $R_\sigma$ whose $\bar{k}$-points are products of $n$ elements of $X^*(\bar{k})$ in $R_\sigma(\bar{k})$. The surface $\bar{X}$ is split, hence it follows from Proposition 3.2 that there exists a unique $c_0 \in R_\sigma(\bar{k})$ such that $P^{r-4}(X^*)(\bar{k}) \subset c_0\mathcal{L}(\bar{k})$. But since $P^{r-4}(X^*)$ and $\mathcal{L}$ are subvarieties of $R_\sigma$, defined over $k$ we conclude that $c_0$ is a $k$-point. If $m$ is $k$-point of $X^*$, then $c_0^{-1}m^{r-4}$ is a $k$-point of $\mathcal{L}$, as required.

To check that the restriction of $U_\sigma \to Y_\sigma$ to $X \subset Y_\sigma$ is a universal torsor we can go over to $\bar{k}$ where it follows from our main theorem in the split case. QED

**Remark** Let $X$ be a del Pezzo surface with a $k$-point, of degree $5, 4, 3$ or $2$. Although $(G/P)_a$ contains some universal $X$-torsor, other universal $X$-torsors of the same type are naturally embedded into certain twists of $(G/P)_a$. Indeed, all torsors
of the same type are obtained from any of them by twisting by the cocycles in $Z^1(k, T_\sigma)$. The natural map $H_\sigma \to T_\sigma$ gives a surjection $H^1(k, H_\sigma) \to H^1(k, T_\sigma)$ since $H^1(k, G_m) = 0$ by Hilbert's theorem 90. Therefore, if $T \subset (G/P)_\alpha$ and $\theta \in Z^1(k, H_\sigma)$, then $T_\theta$ is contained in the twist of $(G/P)_\alpha$ by the 1-cocycle in $Z^1(k, G)$ coming from $\theta \in Z^1(k, H_\sigma)$. By general theory ([14], I.5) this twist is a left homogeneous space of the inner form of $G$ defined by $\theta$. (In the case of Gr(2, 5) we only obtain twists that are isomorphic to Gr(2, 5) because $H^1(k, SL(n)) = 0$.) We note that by Steinberg's theorem every class in $H^1(k, G)$ comes from $H^1(k, H_\sigma)$ for some maximal torus $H_\sigma \subset G$.

**Corollary 4.5** Let $X$ be a del Pezzo surface of degree 4 such that universal $X$-torsors exist. Let $\sigma \in H^1(\Gamma, W)$ be the class defined by the action of the Galois group on the exceptional curves of $X$.

(i) $X^\times$ and $Y_\sigma^\times$ are $k$-subvarieties of $R_\sigma$. Moreover, $Y_\sigma$ contains $cX$ for some $c \in R_\sigma(k)$ if and only if $X$ has a $k$-point.

(ii) If $X \subset Y_\sigma$, then $X = Y_\sigma \cap mY_\sigma$ for some $m \in R_\sigma(k)$.

**Proof.** (i) In view of Theorem 4.4 it remains to prove the ‘only if’ part. We note that $Y_\sigma$ embeds into $R_\sigma$ by Corollary 4.3. The existence of universal $X$-torsors implies that $X$ embeds into $R_\sigma$, as was discussed before Theorem 4.4. For $r = 5$ the inclusion $X^\times \subset c_0L$ from the proof of Theorem 4.4 is an equality by the remark in the end of Section 3. Hence if $L$ has a $k$-point, then so does $X^\times$.

(ii) A del Pezzo surface of degree 4 with a $k$-point is known to be unirational (i.e., is dominated by a $k$-rational variety, see [10]). Thus $X$, and hence also $L = c_0^{-1}X^\times$ contains a Zariski dense set of $k$-points. The variety $X$ is contained in $Y_\sigma \cap h^{-1}c_0Y_\sigma$ for any $h \in X^\times(k)$, and this inclusion is an equality for any $h$ in a dense open subset of $X$, see the remark after the proof of Theorem 2.5. QED

**Remark** By the previous remark an arbitrary universal torsor over a del Pezzo surface $X$ of degree 4 with a $k$-point embeds into a twisted form of $(G/P)_\alpha$ by a cocycle coming from $\theta \in Z^1(k, H_\sigma)$. This twisted form $(G/P)_{\alpha, \theta}$ is naturally a subset of a vector space (non-canonically isomorphic to $V$) acted on by $S_\sigma$. Thus any universal torsor over $X$ is an open subset of the intersection of two $k$-dilatations of $(G/P)_{\alpha, \theta}$.

**References**


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