1. Problems

(1) Let $S$ be a surface with a chart $(\phi, U)$ so that
\[ \frac{\partial \phi}{\partial x_1} \cdot \frac{\partial \phi}{\partial x_2} = 0 \]
for all $(x_1, x_2) \in U$. Show that
\[ K \circ \phi = -\frac{1}{2\sqrt{g_{11}g_{22}}} \left[ \frac{\partial}{\partial x_2} \left( \frac{\partial_{x_2}g_{11}}{\sqrt{g_{11}g_{22}}} \right) + \frac{\partial}{\partial x_1} \left( \frac{\partial_{x_1}g_{22}}{\sqrt{g_{11}g_{22}}} \right) \right], \]
where $g_{11} = \frac{\partial \phi}{\partial x_1} \cdot \frac{\partial \phi}{\partial x_1}$ and $g_{22} = \frac{\partial \phi}{\partial x_2} \cdot \frac{\partial \phi}{\partial x_2}$.

(2) Let $S$ be a compact surface with no boundary and let
\[ P = \{ x \in S : K(x) \geq 0 \}, \]
where $K$ is the gaussian curvature.

a) Show that the Gauss map of $S$ restricted to $P$ is surjective. In other words, for every $p \in S^2 = \{ x^2 + y^2 + z^2 = 1 \}$, there is $x \in P$ so that $N(x) = p$.

b) Show that
\[ \int_S K_+ \, dA \geq 4\pi, \]
where $K_+(x) = \max\{K(x), 0\}$ for all $x \in S$.

HINT: If $N, M$ are two surfaces and $F$ is a surjective smooth map $F : N \to M$, then
\[ \text{area}(M) \leq \int_N |\det DF| \, dA. \]

(3) (Do Carmo, 4.2, Ex 8) Let $G : \mathbb{R}^3 \to \mathbb{R}^3$ be a map such that
\[ |G(p) - G(q)| = |p - q| \quad \text{for all } p, q \in \mathbb{R}^3. \]
Show the existence of $p_0 \in \mathbb{R}^3$ and $B$ a matrix in $O(3)$ such that
\[ G(p) = B(p) + p_0, \quad \text{for all } p \in \mathbb{R}^3. \]

(4) (Do Carmo, 4.2, Ex 11)
(a) Let $S \subset \mathbb{R}^3$ be a regular surface and let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a distance preserving map (like the one in the previous exercise) such that $F(S) \subset S$. Prove that the restriction of $F$ to $S$ is an isometry of $S$.

(c) Use (a) to show that the group of isometries of $\{x^2+y^2+z^2 = 1\}$ is contained in the group of orthogonal linear maps of $\mathbb{R}^3$, which is $O(3)$.

(c) Give an example to show that there are isometries $\phi : S_1 \to S_2$ which cannot be extended into distance preserving maps $F : \mathbb{R}^3 \to \mathbb{R}^3$.

(5) (Do Carmo, 4.2, Ex 12) Let $C = \{x^2 + y^2 = 1\}$. Construct an isometry $\phi : C \to C$ such that the set of fixed points $\{p \in C : \phi(p) = p\}$ contains exactly two points.

(6) (Do Carmo, 4.3, Ex 2) Show that if we have a parametrization of a surface such that $g_{11} = g_{22} = \lambda(x_1, x_2)$ and $g_{12} = 0$ then
$$K = -\frac{1}{2\lambda} \Delta(\log \lambda).$$

Conclude that if $g_{11} = g_{22} = (x_1^2 + x_2^2 + c)^{-2}$, then $K = 4c$, where $c$ is a constant.

(7) (Do Carmo, 4.3, Ex 3) Verify that the surfaces parametrized by
$$\mathbf{x}(u, v) = (u \cos v, u \sin v, \log u)$$
$$\tilde{\mathbf{x}}(u, v) = (u \cos v, u \sin v, \text{volume})$$
have equal Gaussian curvature at points $\mathbf{x}(u, v)$ and $\tilde{\mathbf{x}}(u, v)$ but $\tilde{\mathbf{x}} \circ \mathbf{x}$ is not an isometry. This shows that the converse to Gauss Egregium’s Theorem is not true.

(8) Show that there exist no surface with $g_{11} = g_{22} = 1$ and $A_{11} = 1$, $A_{22} = -1$, and $A_{12} = 0$.

(9) Let $S$ be a compact surface in $\mathbb{R}^3$ with no boundary which has positive Gaussian curvature and denote the Gauss map by
$$N : S \to \{x^2 + y^2 + z^2 = 1\}.$$

Show that if $\gamma$ is a geodesic in $S$ which divides $S$ into a set $A$ and another set $B$ (i.e., $S = A \cup B$ and $\partial A = \partial B = \gamma$), then
$$\text{area}(N(A)) = \text{area}(N(B)).$$

Hint: If you have a diffeomorphism
$$F : S \to \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3,$$
what is the formula for $\text{area}(F(A))$ in terms of $DF$ and the set $A$?
(10) Let $S$ be the cylinder $\{x^2 + y^2 = 1\}$ in $\mathbb{R}^3$, with the chart
\[ \phi : [0, 2\pi] \times \mathbb{R} \longrightarrow S, \quad \phi(\theta, z) = (\cos \theta, \sin \theta, z). \]
and let $\gamma$ be a simple closed curve in $S$.

(a) Let $\gamma$ be a simple closed curve in $S$ for which there is a path
\[ \alpha : [0, 1] \longrightarrow [0, 2\pi] \times \mathbb{R} \]
so that $\alpha(0) = (0, z_0)$, $\alpha(1) = (2\pi, z_0)$, and $\gamma = \phi \circ \alpha$. In other words, $\gamma$ loops once around the $z$-axis. If $\vec{\nu}$ denotes a unit normal vector to $\gamma$ show that
\[ \int_{\gamma} \vec{k} \cdot \vec{\nu} d\sigma = 0. \]

(b) Let $\gamma$ be a simple closed curve in $S$ for which there is a path
\[ \alpha : [0, 1] \longrightarrow [0, 2\pi] \times \mathbb{R} \]
so that $\alpha(0) = (\theta_0, z_0) = \alpha(1)$. In other words, $\gamma$ is the boundary of a disc $D$ in $S$. Show that $\gamma$ is not a geodesic.

(11) (Do Carmo, 4.4, Ex 5 a) and 6) Consider the torus of revolution generated by rotating the circle
\[ (x - a)^2 + z^2 = r_0^2, \quad y = 0 \]
about the $z$-axis ($a > r_0 > 0$). The parallels generated by the points
$(a + r_0, 0), (a - r_0, 0), (a, r_0)$ are called the maximum parallel, the minimum parallel, and the upper parallel, respectively.

(a) Check which of these parallels is a geodesic (i.e., $k_g = 0$).
(b) Compute the geodesic curvature of the upper parallels.

(12) (Do Carmo, 4.4, Ex 7) Intersect the cylinder $\{x^2 + y^2 = 1\}$ with a plane passing through the $x$-axis and making an angle $\theta$ with the $xy$-plane.

(a) Show that the intersecting curve is an ellipse $C$.
(b) Compute the absolute value of the geodesic curvature of $C$ in the cylinder at the point where $C$ meets the axes.

(13) Let $S$ be a surface and suppose there is $F : U \subset \mathbb{R}^2 \longrightarrow S$ so that for every $(t_0, s_0) \in U$ i) the curves $\alpha(t) = F(t, s_0)$ and $\alpha(s) = F(t_0, s)$ are geodesics and ii) $\partial_t F(t_0, s_0), \partial_s F(t_0, s_0) = 0$ for every $s_0, t_0$ fixed. Show that the Gaussian curvature is zero.

(14) Consider the surface $S$ parametrized by
\[ \phi(u, v) = (u \sin(v), -u \cos(v), v), (u, v) \in \mathbb{R}^2. \]
(a) Show that \( \phi \) is a chart, i.e., injective and \( \partial_u \phi, \partial_v \phi \) are linearly independent.

(b) Compute the Gaussian curvature and mean curvature of \( S \).

(15) Show that if \( S \) is a connected surface with nonnegative Gaussian curvature and zero mean curvature, then it must be a plane.

(16) Given a positive function \( r : [a, b] \to (0, +\infty) \) consider the surface of revolution

\[ S = \{(r(s) \cos \theta, r(s) \sin \theta, s) : s \in [a, b], 0 < \theta < 2\pi \}. \]

(a) Make a picture of \( S \) to make sure you understand what it is.

(b) If the function \( r \) has a local minimum at \( s_0 \) with \( r''(s_0) > 0 \), show that the Gaussian curvature of \( S \) at \( p = (r(s_0) \cos \theta, r(s_0) \sin \theta, \theta) \) is negative.

(c) Suppose the Gaussian curvature of \( S \) is nonnegative and

\[ r(s) \leq r(a) = r(b) \]

for all \( s \). Show that \( S \) must be a cylinder.

2. Solutions

(1) During the proof of Gauss’s Egregium Theorem we saw that

\[ K = \frac{\partial_{x_1} (\phi^T_{x_1x_2}) \cdot \phi_{x_1} - \partial_{x_2} (\phi^T_{x_1x_2}) \cdot \phi_{x_2}}{g_{11}g_{22}}, \]

where \( \phi_{x_1} = \partial_{x_1} \phi, \phi_{x_1x_2} = \partial^2_{x_1x_2} \phi \) and so on. The thing we want to show is

\[ K = -\frac{1}{2\sqrt{g_{11}g_{22}}} \left[ \frac{\partial}{\partial x_2} \left( \frac{\partial_{x_2} g_{11}}{\sqrt{g_{11}g_{22}}} \right) + \frac{\partial}{\partial x_1} \left( \frac{\partial_{x_1} g_{22}}{\sqrt{g_{11}g_{22}}} \right) \right]. \]

Let \( N \) denote a unit normal to the surface. We have

\[ \phi^T_{x_1x_2} = \phi_{x_1x_2} - (\phi_{x_1x_2} \cdot N) N \]

and so

\[ \partial_{x_2} (\phi^T_{x_1x_2}) = \phi_{x_1x_2x_1} - \partial_{x_2} (\phi_{x_1x_2} \cdot N) N - (\phi_{x_1x_2} \cdot N) \partial_{x_2} N. \]

Thus, using the fact that \( \partial_{x_1} \phi, \partial_{x_2} \phi \) are orthogonal

\[ \partial_{x_2} (\phi^T_{x_1x_2}) \cdot \phi_{x_1} = \partial_{x_2} (\phi^T_{x_1x_2} \cdot \partial_{x_1}) - \phi^T_{x_1x_2} \cdot (\phi_{x_1} \cdot \partial_{x_1}) = \partial_{x_2} (\phi_{x_1x_2} \cdot \partial_{x_1}) - \phi^T_{x_1x_2} \cdot (\phi_{x_1} \cdot \partial_{x_1}) \]

\[ = \partial_{x_2} (\phi_{x_1x_2} \cdot \partial_{x_1}) - \frac{(\phi_{x_1x_2} \cdot \phi_{x_1}) (\phi_{x_1} \cdot \partial_{x_1})}{g_{11}} - \frac{(\phi_{x_1x_2} \cdot \phi_{x_1}) (\phi_{x_2} \cdot \phi_{x_1x_1})}{g_{22}} \]

\[ = \frac{\partial^2 g_{11}}{2} - \frac{(\partial_{x_2} g_{11})^2}{4g_{11}} - \frac{(\partial_{x_2} g_{22})^2}{4g_{22}}. \]

Now

\[ \phi^T_{x_2x_2} = \phi_{x_1x_2} - (\phi_{x_1x_2} \cdot N) N \]
and so, using again the fact that \( \partial_{x_1} \phi \partial_{x_2} \phi = 0 \), we have

\[
\partial_{x_1}(\phi^T_{x_2x_2}).\phi_{x_1} = \partial_{x_1}(\phi_{x_2x_2}.\phi_{x_1}) - \partial_{x_2}(\phi_{x_2x_1}) = \partial_{x_1} \partial_{x_2}(\phi_{x_2}.\phi_{x_1}) - \partial_{x_1} \phi_{x_2x_2}.\phi_{x_1x_1} \\
= -\partial_{x_1}(\phi_{x_2}.\phi_{x_2x_1}) - \phi^T_{x_2x_2}.\phi_{x_1x_1} = -\partial_{x_1}(\phi_{x_2}.\phi_{x_2x_1}) - (\phi_{x_2x_2}.\phi_{x_1})(\phi_{x_1}.\phi_{x_1x_1}) \frac{g_{11}}{g_{22}} \\
= -\partial_{x_1}(\phi_{x_2}.\phi_{x_2x_1}) + (\phi_{x_1x_2}.\phi_{x_2})(\phi_{x_1}.\phi_{x_1x_1}) + (\phi_{x_2x_2}.\phi_{x_2})(\phi_{x_1}.\phi_{x_1x_1}) \frac{g_{11}}{g_{22}} \\
= \frac{\partial^2 g_{12}}{2} + \frac{\partial_{x_2} g_{22} \partial_{x_1} g_{11}}{4g_{22}} + \frac{\partial_{x_2} g_{22} \partial_{x_1} g_{11}}{4g_{11}}.
\]

From this last two identities direct computation shows that

\[
\frac{\partial_{x_1}(\phi^T_{x_2x_2}).\phi_{x_1} - \partial_{x_2}(\phi^T_{x_1x_2}).\phi_{x_1}}{g_{11}g_{22}} = -\frac{1}{2\sqrt{g_{11}g_{22}}} \left[ \frac{\partial}{\partial x_2} \left( \frac{\partial_{x_2} g_{11}}{\sqrt{g_{11}g_{22}}} \right) + \frac{\partial}{\partial x_1} \left( \frac{\partial_{x_1} g_{22}}{\sqrt{g_{11}g_{22}}} \right) \right].
\]

(2) Let’s solve (a) first. Choose \( N \) to be the exterior unit normal and this part is important because if \( N \) were the interior unit normal the argument would need to have some (trivial) changes. Choose \( p \in S^2 \) we want to find \( x \in S \) with \( N(x) = p \).

Let

\[
P_t = \{ \bar{x} \in \mathbb{R}^3 : p,x = t \}.
\]

For \( t \) very large and positive we have \( P_t \cap S = \emptyset \) and clearly there is some \( s \) so that \( P_s \cap S \) is not empty. So we can consider

\[t_0 = \inf\{t : P_s \cap S = \emptyset \text{ for all } s \geq t\}.\]

We have \( P_{t_0} \cap S \neq \emptyset \), otherwise we could find \( \varepsilon \) small so that \( P_{t_0-\varepsilon} \cap S = \emptyset \) and this contradicts the minimality property of \( t_0 \).

Let \( x \in P_{t_0} \cap S \). We want to argue that \( N(x) = p \). From the graphical decomposition lemma we know that in a neighborhood of \( x \) we can write \( S \) as being the graph of a function defined on a set of \( T_x S \). More rigorously, there is an open set \( U \subset T_x S \), an open neighborhood \( V \) of \( x \) in \( \mathbb{R}^3 \), and a function \( f : U \to \mathbb{R} \) so that

\[S \cap V = \{ f(y)N(x) + y + x : y \in U \}.\]

The function \( f \) has \( f(0,0) = 0 \) and \( \nabla f(0,0) = 0 \) necessarily (why?).

Now we must have \( z,p \leq t_0 \) for all \( z \in S \) and \( x,p = t_0 \) (why?). Thus using the graphical parametrization

\[t_0 \geq (f(y)N(x) + y + x).p = f(y)N(x).p + y.p + t_0.\]

In other words \( g(y) = f(y)N(x).p + y.p + t_0 \) has a global maximum at \( y = 0 \) and, because \( \nabla f(0,0) = 0 \), this implies \( p \) is orthogonal to \( T_x S \) (because \( \partial_x g(0,0) = \partial_y g(0,0) = 0 \)).

The fact that \( N \) was the exterior unit normal implies \( N(x) = p \) instead of \( N(x) = -p \).
We just need to conclude that $K(x) \geq 0$. Well, by what we have done we have $T_\varepsilon S + s = P_{t_0}$ and $S$ lies all in one side of $P_{t_0}$. Thus if $K(x) < 0$, the surface would lie on both sides of $P_{t_0}$, which is impossible.

For part (b) consider $L_\varepsilon = \{ x \in S : K(x) > -\varepsilon \}$ which is a surface as well (the set $\{ x : K(x) \geq 0 \}$ is not surface strictus sensus). Restrict the Gauss map to $N : L_\varepsilon \to S^2$ which is surjective. Then by the hint

$$4\pi \leq \int_{L_\varepsilon} |\det N| dA = \int_{L_\varepsilon} |K| dA = \int_{L_0} K dA + \int_{L_\varepsilon \setminus L_0} |K| dA \leq \int_{L_0} K dA + \varepsilon \text{area}(S) = \int_S K_+ dA + \varepsilon \text{area}(S).$$

Making $\varepsilon$ tend to zero the result follows.

(3) The first step is to show that $G$ is a linear map. Fix a point $p$ and consider $\alpha_1(t) = p + tX$, where $X$ is any vector in $\mathbb{R}^3$. Set $F(t) = G \circ \alpha_1(t) - G(q)$. By differentiation in the $t$ variable when $t = 0$ we have

$$|F(t)|^2 = |tX + p - q|^2 \implies F'(0).F(0) = X.(p - q)$$

$$\implies DG_p(X).(G(p) - G(q)) = X.(p - q) \implies X.(DG_p)^\top (G(p) - G(q)) = X.(p - q)$$

for all $X$ and all $q$. In the last line we use the fact that if $A$ is a $3 \times 3$ matrix with transpose $A^T$, then $A(X).Y = X.A^T(Y)$ for every vectors $X,Y$. Because the identity we derived is valid for all $X$ it is simple to conclude that

$$(DG_p)^\top (G(p) - G(q)) = p - q \quad \text{for every } q \in \mathbb{R}^3.$$ 

In particular the linear map $(DG_p)^T$ is surjective and thus injective. Denoting by $B$ its inverse we have from the above formula that

$$G(p) = G(q) + B(p - q) \text{for every } p, q \in \mathbb{R}^3.$$ 

This show that $G$ is a linear map (just take $p_0 = G(0) - B(0)$ for some $q$ fixed.)

Using again the hypothesis we obtain that

$$|B(p - q)| = |G(p) - G(q)| = |p - q|.$$ 

Thus $B$ preserves distances and so it is an isometry of $\mathbb{R}^3$ as well. It is easy to conclude that $B \in O(3)$ by differentiating the identity below and set $t = 0$

$$B(X + tY).B(X + tY) = |X + tY|^2 = |X|^2 + 2tX.Y + t^2|Y|^2.$$ 

(4) First a). From the previous exercise we know that $F$ is a linear isometry of $\mathbb{R}^3$, which means that $DF_p = DF_0$ for all $p$ in $\mathbb{R}^3$ (i.e. $F$ is linear) and $DF_0(X).DF_0(Y) = X.Y$ for all vectors $X,Y$ in
\(\mathbb{R}^3\). In particular for every \(p \in S\) we have that \(DF_p = DF_0\) and \(DF_p(X).DF_p(Y) = X.Y\) for all \(X, Y\) in \(T_pS\), which means that \(F\) is an isometry of \(S\). Technically speaking one should show that if \(F\) is an ambient map of \(\mathbb{R}^3\), then the restriction of \(F\) to \(S\) has the property that for every \(X \in T_pS\), then \(DF_p(X)\) as it was defined in class is nothing but \(DF_p(X)\) as it was defined in multivariable calculus. This is just the chain rule and so I will not say anything else to avoid the risk of just ending up making it more confusing.

Now b). The orthogonal linear transformations are just those 3×3 matrices which have \(A^T A = Id\), i.e., \(A^{-1} = A^T\). In this case we have

\[|A(X)|^2 = A(X).A(x) = X.A^T A(X) = X.X = |X|^2\]

and so \(A\) send the unit sphere into the unit sphere. Moreover \(|A(X) - A(Y)| = |A(X - Y)| = |X - Y|\), and so its distance preserving in the sense of Exercise 8. Thus a) can be applied to conclude \(A\) is an isometry of the sphere.

Finally c). Take \(F(x, y, 0) = (\cos x, \sin x, y)\) which is an isometry from part of the plane \(\{z = 0\}\) into part of the cylinder \(\{x^2 + y^2 = 1\}\). \(F\) is not distance preserving because \(|F(0, 0, 0) - F(\theta, 0, 0)| = 2(1 - \cos \theta)\) and \(|(0, 0, 0) - (\theta, 0, 0)| = |\theta|\).

(5) Consider \(F(x, y, z) = (x, -y, -z)\). This is an isometry of the cylinder \(S = \{x^2 + y^2 = 1\}\) because \(F\) send the \(S\) into \(S\) and \(F\) is an orthogonal linear transformation. Finally \(F(x, y, z) = (x, y, z)\) then \(z = 0\) and \(y = 0\). But \(x^2 + y^2 = 1\), which means \(x = \pm 1\), i.e., the only fixed points are \((1, 0, 0)\) and \((-1, 0, 0)\).

(6) Let’s use exercise 1. Set \(\lambda = e^{2u}\) for convenience. Then we have

\[\sqrt{g_1 g_2} = e^{2u}, \quad \frac{\partial x_1 g_{11}}{(g_1 g_2)^{1/2}} = 2\partial x_2 u, \quad \frac{\partial x_2 g_{22}}{(g_1 g_2)^{1/2}} = 2\partial x_1 u.\]

Thus, recalling that \(\Delta h = \partial x_1 x_1 h + \partial x_2 x_2 h\), we obtain

\[K = -e^{-2u} \Delta u = -\frac{1}{2\lambda} \Delta \ln \lambda.\]

(Recall that \(\ln \sqrt{\lambda} = \frac{1}{2} \ln \lambda\).

To be consistent with the rest of the notation I should use \(\lambda = (x_1^2 + x_2^2 + c)^{-2}\) for the last part. Then

\[\partial x_i \ln \lambda = -2 \partial x_i \ln (x_1^2 + x_2^2 + c) = -\frac{4x_i}{x_1^2 + x_2^2 + c}\]

and

\[\partial^2 x_i x_i \ln \lambda = -\frac{4}{x_1^2 + x_2^2 + c} + \frac{8x_i^2}{(x_1^2 + x_2^2 + c)^2}.\]
Thus

$$\Delta \ln \lambda = -\frac{8}{x_1^2 + x_2^2 + c} + \frac{8(x_1^2 + x_2^2)}{(x_1^2 + x_2^2 + c)^2} = \frac{-8c}{(x_1^2 + x_2^2 + c)^2} = -8c \lambda$$

and so

$$K = -\frac{1}{2\lambda} \Delta \ln \lambda = 4c.$$

(7) We have

$$\frac{\partial x}{\partial u} = (\cos v, \sin v, u^{-1}), \quad \frac{\partial x}{\partial v} = (-u \sin v, u \cos v, 0)$$

and so

$$g_{11} = 1 + u^{-2}, g_{12} = 0, g_{22} = u^2.$$  

Because $\partial_v g_{11} = 0$ we obtain from Exercise 1

$$\bar{K} = -\frac{1}{2\sqrt{1+u^2}} \frac{\partial}{\partial u} \left( \frac{2u}{\sqrt{1+u^2}} \right) = -\frac{1}{(1+u^2)^2}$$

We have

$$\frac{\partial \bar{x}}{\partial u} = (\cos v, \sin v, 0), \quad \frac{\partial \bar{x}}{\partial v} = (-u \sin v, u \cos v, 0)$$

and so

$$\bar{g}_{11} = 1, \bar{g}_{12} = 0, \bar{g}_{22} = 1 + u^2.$$  

Using the formula in Exercise 1 we have

$$\bar{K} = -\frac{1}{2\sqrt{1+u^2}} \frac{\partial}{\partial u} \left( \frac{2u}{\sqrt{1+u^2}} \right) = -\frac{1}{(1+u^2)^2}.$$  

Therefore the surfaces have the same curvature.

If the map $\bar{x} \circ \bar{x}^{-1}$ were an isometry then by Proposition 1, page 220, we would have $g_{11} = \bar{g}_{11}, g_{12} = \bar{g}_{12},$ and $g_{22} = \bar{g}_{22}$. Because this is false the surfaces are not isometric.

(8) If there was such a surface then from the definition of Gaussian curvature we would have $K = (A_{11}A_{22} - A_{12}^2)(g_{11}g_{22} - g_{12}^2)^{-1} = -1$.  

But from Gauss’s Theorem we know that $K$ is intrinsic, i.e., depends only on $g_{11}, g_{12}$ and $g_{12}$. Thus $K$ must be zero because we can surely put coordinates on a plane which have $g_{11} = g_{22} = 1, g_{12} = 0$ and a plane has Gaussian curvature zero.

(9) Because $K > 0$, we have from Gauss-Bonnet that $S$ is diffeomorphic to a sphere. Thus $A$ and $B$ are both diffeomorphic to a disc and
\( \partial A = \partial B = \gamma \) is a geodesic. Hence, if \( \nu \) denotes the interior unit normal to \( \partial A \) we have

\[
\int_A K dA + \int_{\partial A} k \nu d\sigma = 2\pi \implies \int_A K dA = 2\pi.
\]

Likewise

\[
\int_B K dA = 2\pi.
\]

I will show now that

\[
\text{area}(N(A)) = \int_A K dA = 2\pi
\]

and this completes the exercise.

The first remark is that the Gaussian map \( N \) must be a diffeomorphism. Why? Well, because \( K = \det A = -\det dN \), we have from the inverse function theorem that \( N \) is locally a diffeomorphism. And it happens that locally diffeomorphisms from a sphere into a sphere must be global diffeomorphism.

The second remark is to show the hint, namely that

\[
\text{area}(F(A)) = \int_A |\det dF| dA.
\]

Suppose that \( \phi : D \subset \mathbb{R}^2 \to A \) is a chart for \( A \). Then \( \psi = F \circ \phi \) is a chart for \( N = \{ x^2 + y^2 + z^2 = 1 \} \) because \( F \) is a diffeomorphism.

From the lectures we know that

\[
\text{area}(\psi(D)) = \int_D \left| \frac{\partial \psi}{\partial x} \times \frac{\partial \psi}{\partial y} \right| dxdy.
\]

Now let \( \{ e_1, e_2 \} \) be a basis for \( T_{\phi(p)}A \) which has \( dF_p(e_1) = \lambda_1 e_1 \) and \( dF_p(e_2) = \lambda_2 e_2 \), i.e., \( \{ e_1, e_2 \} \) is an eigenbasis for \( dF \). Then

\[
\frac{\partial \phi}{\partial x}(p) = a_1 e_1 + b_1 e_2 \quad \frac{\partial \phi}{\partial y}(p) = a_2 e_1 + b_2 e_2
\]

and thus

\[
\left| \frac{\partial \phi}{\partial x} \times \frac{\partial \phi}{\partial y} \right| = |a_1 b_2 - a_2 b_1||e_1 \times e_2|.
\]

Furthermore

\[
\frac{\partial \psi}{\partial x}(p) = dF_{\phi(p)} \left( \frac{\partial \phi}{\partial x}(p) \right) = \lambda_1 a_1 e_1 + \lambda_2 b_1 e_2
\]

and

\[
\frac{\partial \psi}{\partial y}(p) = dF_{\phi(p)} \left( \frac{\partial \phi}{\partial y}(p) \right) = \lambda_1 a_2 e_1 + \lambda_2 b_2 e_2.
\]

Thus

\[
\left| \frac{\partial \psi}{\partial x} \times \frac{\partial \psi}{\partial y} \right| = |\lambda_1 \lambda_2||a_1 b_2 - a_2 b_1||e_1 \times e_2| = |\det dF_{\phi(p)}| \left| \frac{\partial \phi}{\partial x} \times \frac{\partial \phi}{\partial y} \right|.
\]
(This formula was long to derive but it should be highly expected). Therefore we have

\[
\text{area}(F(\phi(D))) = \text{area}(\psi(D)) = \int_D \left| \frac{\partial \psi}{\partial x} \times \frac{\partial \psi}{\partial y} \right| dxdy.
\]

\[
= \int_D |\det F_{\phi(D)}| \left| \frac{\partial \phi}{\partial x} \times \frac{\partial \phi}{\partial y} \right| dxdy = \int_{\phi(D)} |\det dF|dA.
\]

Covering \(A\) with a finite number of charts we arrive at

\[
\text{area}(F(A)) = \int_A |\det dF|dA.
\]

Using this formula with \(F = N\) the Gauss map we obtain

\[
\text{area}(N(A)) = \int_A |\det dN|dA = \int_A |K|dA = \int_A KdA = 2\pi.
\]

(10) If the path \(\alpha(t) = (\theta(t), z(t))\), choose \(z_1\) so that \(z(t) > z_1\) for all \(t \in [0, 1]\) and consider the path \(\beta(t) = (2\pi t, z_1)\). Note that \(\phi \circ \beta\) is a geodesic and there is a region \(R\) in \(S\) so that \(\partial S = \gamma \cup \phi \circ \beta\). To recognize this last fact just let \(R = \phi(A)\), where \(A\) is the region in \([0, 2\pi] \times \mathbb{R}\) below \(\alpha\) and above \(\beta\). Note that \(R\) is dieromorphic to a cylinder and thus has zero Euler characteristic. Using Gauss-Bonnet we obtain that

\[
\int_R KdA + \int_\gamma \bar{k} \bar{\nu}d\sigma + \int_{\phi \circ \beta} \bar{k} \bar{\nu}d\sigma = 0,
\]

where \(\bar{\nu}\) is the interior unit normal. Because \(K = 0\) and \(\phi \circ \beta\) is a geodesic we obtain that

\[
\int_\gamma \bar{k} \bar{\nu}d\sigma = 0,
\]

which is what we wanted to show.

For the second one note that the Euler characteristic of \(D\) is one and so by the Gauss-Bonnet Theorem we have

\[
\int_D KdA + \int_\gamma \bar{k} \bar{\nu}d\sigma = 2\pi \implies \int_\gamma \bar{k} \bar{\nu}d\sigma = 2\pi.
\]

Thus, \(\gamma\) cannot be a geodesic.

(11) Consider a parametrization of \(S\)

\[
\phi(s, \theta) = (r(s) \cos \theta, r(s) \sin \theta, z(s)).
\]

Then

\[
\frac{\partial \phi}{\partial s} = (r' \cos \theta, r' \sin \theta, z'), \quad \frac{\partial \phi}{\partial \theta} = (-r \sin \theta, r \cos \theta, 0),
\]
\[
\frac{\partial^2 \phi}{(\partial s)^2} = (r'' \cos \theta, r'' \sin \theta, z''), \quad \frac{\partial^2 \phi}{(\partial \theta)^2} = (-r \cos \theta, -r \sin \theta, 0),
\]

\[
\frac{\partial^2 \phi}{\partial \theta \partial s} = (-r' \sin \theta, r' \cos \theta, 0).
\]

and so

\[
g_{11} = (r')^2 + (z')^2 = 1, \quad g_{12} = 0, \quad g_{22} = r^2,
\]

where I am assuming, without loss of generality, that \((r(s), z(s))\) is parametrized by arc-length. If \(\alpha(t) = \phi(s(t), \theta(t))\) is a curve parametrized by arc-length then, denoting differentiation with respect to \(t\) with a dot, we have

\[
\ddot{\alpha} = \frac{d}{dt}(\dot{s}\partial_s \phi + \dot{\theta}\partial_\theta \phi) = \ddot{s}\partial_s \phi + \ddot{\theta}\partial_\theta \phi + (\dot{s})^2 \partial_{ss}^2 \phi + 2\dot{s}\dot{\theta}\partial_{s\theta} \phi + (\dot{\theta})^2 \partial_{\theta\theta} \phi.
\]

Consider the orthonormal basis \(X_1, X_2\) for the tangent plane of \(S\) where

\[
X_1 = \frac{\partial \phi}{\partial s}, \quad X_2 = r^{-1} \frac{\partial \phi}{\partial \theta} = (-\sin \theta, \cos \theta, 0).
\]

We have

\[
\ddot{\alpha}.X_1 = \ddot{s} + (\dot{s})^2 (r''r' + z''z') - (\dot{\theta})^2 rr' = \ddot{s} - (\dot{\theta})^2 rr',
\]

where we use the fact that

\[
(r')^2 + (z')^2 = 1 \implies r''r' + z''z' = 0,
\]

and

\[
\ddot{\alpha}.X_2 = \ddot{\theta} + 2\dot{s}\dot{\theta}rr'.
\]

The normal vector \(N\) is given by \(N = X_1 \times X_2\) and \(E = N \times \ddot{\alpha}\) is given by

\[
E = N \times (\dot{s}\partial_s \phi + \dot{\theta}\partial_\theta \phi) = N \times (\dot{s}X_1 + \dot{\theta}rX_2)
\]

\[
= (X_1 \times X_2) \times (\dot{s}X_1 + \dot{\theta}rX_2) = \ddot{s}X_2 - \dot{\theta}rX_1.
\]

Therefore

\[
k_g = \ddot{\alpha}.E = \ddot{s}(\ddot{\theta} + 2\dot{s}\dot{\theta}rr') - \dot{\theta}(\ddot{s} - (\dot{\theta})^2 rr')
\]

For the case in point we have

\[
r(s) = a + r_0 \cos(s/r_0), \quad z(s) = r_0 \sin(s/r_0), \quad \text{where } (r')^2 + (z')^2 = 1.
\]

The curves are

- maximum parallel: \(\alpha_1(t) = \phi(0, t/r_0)\),
- minimum parallel: \(\alpha_2(t) = \phi(r_0\pi, t/r_0)\),
- upper parallel: \(\alpha_3(t) = \phi(r_0\pi/2, t/r_0)\)
I use the term $t/r_0$ instead of $t$ because I want the curves to be parametrized by arc-length. In any case we have $\theta = r_0^{-1}, \dot{s} = \ddot{\theta} = 0,$ and $r' = -\sin(s/r_0)$. Using the formula above for $\vec{k}$ we get

$$k_g(\alpha_1) = r_0^{-3} r(0) r'(0) = 0$$

$$k_g(\alpha_3) = r_0^{-3} r(r_0 \pi) r'(r_0 \pi) = 0$$

and

$$k_g(\alpha_3) = r_0^{-3} r(r_0 \pi/2) r'(r_0 \pi/2) = \frac{a}{r_0^2}.$$ 

Note that $k_g(\alpha_3)$ compute the geodesic curvature of the upper parallels.

(12) We use the coordinate chart

$$\phi(\text{std}, z) = (\cos s, \sin s, z)$$

which makes it into a local isometry with Euclidean plane, i.e., $g_{11} = 1 = g_{22}, F = 0$. The plane which intersects the cylinder is $P = \{z = y \tan \theta\}$ and so the curve $C$ is parametrized by $C(t) = (\cos t, \sin t, \sin t \tan \theta) = \phi(t, \sin t \tan \theta)$. If $e_1 = (1, 0, 0), e_2 = (0, \cos \theta, \sin \theta)$ is an orthonormal basis for $P$, then

$$C = \{xe_1 + ye_2 | x^2 + \cos^2 \theta y^2 = 1\}$$

and so we see that $C$ is indeed an ellipse in the plane $P$. To compute its curvature vector we have the little annoying thing that $C(t)$ is not a parametrization by arc-length. Suppose then that $t = t(u)$ is a change of variable which makes $C(u)$ parametrized by arc length. Then, if $E$ denotes a unit vector lying in the tangent plane to to the cylinder which is orthogonal to $C''(t)$, we know that $k_g = \vec{k}.E$. Now

$$\frac{dC}{du} = C'(t) \frac{dt}{du} \implies \frac{dt}{du} = |C'(t)|^{-1}$$

and thus

$$k_g = \frac{d}{du} \left( C'(t) \frac{dt}{du} \right).E = \left( \frac{dt}{du} \right)^2 C''(t).E + \frac{d^2 t}{(du)^2} C'(t).E = \left( \frac{dt}{du} \right)^2 C''(t).E$$

$$= |C'(t)|^{-2} C''(t).E.$$ 

Let’s use this formula. $C$ intersects the axis when $t = 0, \pi$. In this case $C''(t) = -C(t)$ and so $C''(0) = -(1, 0, 0), C''(\pi) = (1, 0, 0)$. In both cases $C''$ has the direction of the normal vector to the cylinder and so the term $C''(t).E$ is zero because $E$ lies in the tangent plane of the cylinder. Thus $k_g = 0$ at those points.
(13) One obvious consequence of i) is that \( \partial_s F \) and \( \partial_t F \) are never zero. Hence from ii) we get that \( \partial_s F(t, s), \partial_t F(t, s) \) are linearly independent which means that for every \((t_0, s_0) \in U\), there is a small neighborhood \( V \) so that \( F \) restricted to \( V \) is a chart. Therefore we can use the formula from the first exercise in this problem sheet.

From problem 1) it suffices to see that \( \partial_s g_{11} = \partial_t g_{22} = 0 \). Condition ii) is saying that \( g_{12} = 0 \).

The fact that \( t \mapsto F(t, s) \) is a geodesic in \( S \) implies that \( \partial_{tt}^2 F. E = 0 \), where \( E \) is a unit vector lying in the tangent plane of \( S \) and perpendicular to \( \partial_t F \) (the tangent vector to the curve). Thus, because \( g_{12} = 0 \) we have that \( E \) is a multiple of \( \partial_s F \) and so \( \partial_{tt}^2 F. \partial_s F = 0 \).

Therefore

\[
\partial_s g_{11} = 2 \partial_{st}^2 F. \partial_t F = 0.
\]

Same reasoning shows that \( \partial_t g_{22} = 0 \).

(14) (a) \( \phi \) is clearly smooth. Suppose \( \phi(u, v) = \phi(u', v') \). Then we must have \( v = v' \) by looking at the third component of \( \phi \). Thus \( \sin(v) = \sin(v') \) and \( \cos(v) = \cos(v') \). If \( \sin(v) \neq 0 \) we conclude that \( u = u' \) by looking at the first component of \( \phi \). If \( \sin(v) = 0 \) then \( \cos(v) \neq 0 \) and so we conclude that \( u = u' \) by looking at the second component of \( \phi \).

Note that

\[
\frac{\partial \phi}{\partial u} = (\sin v, -\cos v, 0), \quad \frac{\partial \phi}{\partial v} = (u \cos v, u \sin v, 1)
\]

and thus \( \partial_u \phi, \partial_v \phi = 0 \). Because \( \partial_u \phi, \partial_v \phi \) are never zero, this implies they must be linearly independent if they are orthogonal.

(b) The matrix that represents the metric \( g \) is given by

\[
\begin{bmatrix}
g_{11} & g_{12} \\
g_{21} & g_{22}
\end{bmatrix} =
\begin{bmatrix}
1 & 0 \\
0 & 1 + u^2
\end{bmatrix}.
\]

The unit normal vector is

\[
N = \frac{\partial_u \phi \times \partial_v \phi}{|\partial_u \phi \times \partial_v \phi|} = \frac{(-\cos v, -\sin v, u)}{\sqrt{1 + u^2}}.
\]

We have

\[
\partial_{uu}^2 \phi = (0, 0, 0), \quad \partial_{uv}^2 \phi = (\cos v, \sin v, 0), \quad \partial_{vv}^2 \phi = (-u \sin v, u \cos v, 0)
\]

and so the matrix that represents the second fundamental form \( A \) is

\[
\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix} =
\frac{1}{\sqrt{1 + u^2}}
\begin{bmatrix}
0 & -1 \\
-1 & 0
\end{bmatrix}.
\]
Therefore, the matrix that represents $-dN$ in the basis $\partial_u \phi, \partial_v \phi$ is given by

$$\sigma = g^{-1}A = \frac{1}{\sqrt{1 + u^2}} \begin{bmatrix} 1 & 0 \\ 0 & (1 + u^2)^{-1} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \frac{1}{\sqrt{1 + u^2}} \begin{bmatrix} 0 & -1 \\ -(1 + u^2)^{-1} & 0 \end{bmatrix}.$$

Hence

$$K = \det \sigma = -(1 + u^2)^{-3/2}, \quad H = \frac{1}{2} tr \sigma = 0.$$

(15) If $H = 0$, then $\lambda_1 + \lambda_2 = 0$, which means $\lambda_1 = -\lambda_2$. Thus the Gaussian curvature becomes $K = \lambda_1 \lambda_2 = -\lambda_1^2 \leq 0$. The condition $K \geq 0$ implies then that $K = -\lambda_1^2 = 0$, which means $\lambda_1 = \lambda_2 = 0$. Therefore, the surface is umbilic and by a Theorem in the notes we know that it must be contained in a plane (the case of being contained in a sphere it is impossible because $K = 0$.)

(16) (b) We have

$$\partial_r \phi = (r'(s) \cos \theta, r'(s) \sin \theta, 1), \quad \partial_\theta \phi = (-r(s) \sin \theta, r(s) \cos \theta, 0)$$

and so the matrix that represents the metric $g$ is given by

$$\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} (r')^2 + 1 & 0 \\ 0 & r^2 \end{bmatrix}.$$

The normal vector is given by

$$N = \frac{\partial_r \phi \times \partial_\theta \phi}{|\partial_r \phi \times \partial_\theta \phi|} = \frac{(- \cos \theta, - \sin \theta, r')}{\sqrt{1 + (r')^2}}.$$

Finally,

$$\partial_{rr} \phi = (r'' \cos \theta, r'' \sin \theta, 0), \quad \partial_r \theta \phi = (-r' \sin \theta, r' \cos \theta, 0)$$

$$\partial_\theta \theta \phi = (-r \cos \theta, -r \sin \theta, 0)$$

and so the matrix that represents the second fundamental form $A$ is

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \frac{1}{\sqrt{1 + (r')^2}} \begin{bmatrix} -r'' & 0 \\ 0 & r \end{bmatrix}.$$

The matrix that represents $-dN$ is given by

$$\sigma = g^{-1}A = \frac{1}{\sqrt{1 + (r')^2}} \begin{bmatrix} (r')^2 + 1 & 0 \\ 0 & r^2 \end{bmatrix}^{-1} \begin{bmatrix} -r'' & 0 \\ 0 & r \end{bmatrix} = \frac{1}{\sqrt{1 + (r')^2}} \begin{bmatrix} -(r')^2 + 1 & -r'' \\ 0 & r^{-1} \end{bmatrix}.$$
Therefore

\[ K = -(r')^2 + 1)^{-3/2}r^{-1}r''. \]

At a local minimum with \( r''(s_0) > 0 \) we have

\[ K = -\frac{r''(s_0)}{r(s_0)} < 0. \]

- If the Gaussian curvature is nonnegative we have

\[ K = -(r')^2 + 1)^{-3/2}r^{-1}r'' \geq 0 \implies r'' \leq 0. \]

Thus \( r \) is a concave function with \( r(a) = r(b) \) and this implies that \( r(s) = r(a) \) for all \( a \leq s \leq b \). A quick way to see this is to note that \( r'' \leq 0 \) implies \( r' \) is nonincreasing, which means \( r'(a) \geq r'(s) \geq r'(b) \). But \( r(s) \leq r(a) \) and \( r(s) \leq r(b) \) implies \( r'(a) \leq 0 \) and \( r'(b) \geq 0 \). Therefore \( r'(s) = 0 \) for all \( s \) and this means \( r \) is constant.