Some aspects of post-crisis equities modelling

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Outline

Volatility-target indices
  Intro
  Mechanics of Volatility Control
  Model dependence
  Hedging

A Regulatory Modelling challenge

References
Vol target structures find favour post-UK budget (Derivatives Week, 6/6/14)

“Sellside firms are seeing increased appetite for structured products with volatility target mechanisms as a way of providing a primary source of income for UK individuals during retirement. The spike in interest comes on the back of Chancellor George Osborne’s 2014 budget that set out greater investment flexibility for individuals at retirement from April 2015.”

Related studies

What has changed post-crisis?

- myriad of tailored topical indices with built-in risk controls
- clients willing/not willing to take particular risks (e.g. volatility)
- a typical pre-crisis high-flier: straight-out exposure to volatility
Intro

Investment strategies are self-financing

\[ I_t = \sum_{i=1}^{m} h^{(i)}_t A^{(i)}_t, \quad \alpha^{(i)}_t := \frac{h^{(i)}_t A^{(i)}_t}{I_t} \]

\[ dl_t = \sum_{i=1}^{m} h^{(i)}_t dA^{(i)}_t \quad dl_t/I_t = \sum_{i=1}^{m} \alpha^{(i)}_t dA^{(i)}_t / A^{(i)}_t \]

discrete case (\( \Delta Z_n := Z_n - Z_{n-1} \))

\[ I_n = \sum_{i=1}^{m} h^{(i)}_n A^{(i)}_n \quad \alpha^{(i)}_n := \frac{h^{(i)}_n A^{(i)}_n}{I_n} \]

\[ \Delta I_n = \sum_{i=1}^{m} h^{(i)}_{n-1} \Delta A^{(i)}_n \quad I_n/I_{n-1} = \sum_{i=1}^{m} \alpha^{(i)}_{n-1} A^{(i)}_n / A^{(i)}_{n-1} \]
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Intro

- Assume $A^{(1)}$ is the risky asset and $A^{(2)}$ is the cash account,
  $R_n^{(i)} = A_n^{(i)} / A_{n-1}^{(i)} - 1$

- **Total Return** indices evolve according to
  $$I_{n+1} = I_n \left( 1 + \alpha_n^{(1)} R_{n+1}^{(1)} + (1 - \alpha_n^{(1)}) R_{n+1}^{(2)} \right)$$

- this is just the definition of self-financing, rewrite
  $$I_{n+1} = I_n \left( 1 + \alpha_n^{(1)} R_{n+1}^{(1)} - \alpha_n^{(1)} R_{n+1}^{(2)} \right) + I_n R_{n+1}^{(2)}$$

- **Excess Return** index

- both versions are offered, with Excess Return having a bigger share
Mechanics of Volatility Control

single-asset \textit{volatility controlled} investment strategies

\begin{itemize}
  \item $\alpha_t^{(1)}$ — investment in a risky asset — function of asset volatility
  \item $\alpha_t^{(2)} \equiv 1 - \alpha_t^{(1)}$ — investment in cash
\end{itemize}

volatility estimation

\begin{itemize}
  \item simple variance estimator
    \begin{align*}
      v_n^w &= \frac{252}{w - 1} \sum_{j=n-w+1}^{n} \left( R_j^{(1)} - \bar{R}_n \right)^2 \\
      \bar{R}_n &= \frac{1}{w} \sum_{k=n-w+1}^{n} R_k^{(1)}
    \end{align*}
    \begin{itemize}
      \item volatility estimated as $\hat{\sigma}_n^w := \sqrt{v_n^w}$
      \item exponential forgetting is often used to enhance accuracy
    \end{itemize}
\end{itemize}
Crux of the matter

- Assume lognormal asset with vol $\sigma_1$
- Also assume that risky asset is total-return, and can be funded at overnight rate $r$
- Portfolio for target volatility $\sigma_{TV}$: $\alpha_t^{(1)} = \sigma_{TV}/\sigma_1$, $\alpha_t^{(2)} = 1 - \alpha_t^{(1)}$

\[
\frac{dl_t}{l_t} = \frac{\sigma_{TV}}{\sigma_1} \frac{dA_t^{(1)}}{A_t^{(1)}} + \left(1 - \frac{\sigma_{TV}}{\sigma_1}\right) \frac{dA_t^{(2)}}{A_t^{(2)}} = \frac{\sigma_{TV}}{\sigma_1} (r \ dt + \sigma_1 \ dW_t) + \left(1 - \frac{\sigma_{TV}}{\sigma_1}\right) r \ dt = \sigma_{TV} \ dW_t + r \ dt
\]

- $l_t$ is lognormal with volatility $\sigma_{TV}$
- Practitioners often use $\sigma_{TV} + \epsilon$ in Black-Scholes formula when pricing options on $l$
Back to reality

- contractual vol target index looks more like this

\[ l_{n+1} = l_n \left( 1 + \alpha_n^{(1)} R_{n+1}^{(1)} + (1 - \alpha_n^{(1)}) R_{n+1}^{(2)} \right) \]

- participation in the risky asset is capped

\[ \alpha_n^{(1)} = \min(\text{cap}, \sigma_{TV}/\sqrt{v_n^w}) \]

- vol estimation
  - the size of averaging window \( w \) varies from 20 to 60 days
  - frequently estimator is defined as max of two estimators with different \( w \)
  - there are products in which VIX is used instead of the estimator

- underlyings vary across asset classes
- often rebalancing is triggered only when moves in the market are above a threshold
**discrete vs. continuous**

- the contractual version of vol target does not arise from discretization in the usual way

- for corl ol $g$, semimartingale $X$ and partition $0 = t_1^n, t_2^n, \ldots, t_n^n = T$

\[ Y_t^n := \Pi_k \left( 1 + g_{t_k^n}(X_{t^\wedge t_{k+1}^n} - X_{t^\wedge t_k^n}) \right) \]

converges in law on $[0, T]$ to Doléans-Dade exponential $\mathcal{E}(\int_0^t g_u^- dX_u)_t$ as $n \to \infty$ (Emery (1978))

- ties in with the notion of *multiplicative stochastic integral* developed in ’60s ’70s (McKean, Emery,...)
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Master Formula

- for the general case of semimartinglae risky asset $A_{t}^{(1)}$, put $g_{t} := \frac{\sigma_{TV}A_{t}^{(2)}}{\sqrt{\nu_{t}^{w}A_{t}^{(1)}}}$, the vol target strategy converges to

$$e^{rt} \mathcal{E} \left( \int_{0}^{t} \frac{\sigma_{TV}A_{u}^{(2)}}{\sqrt{\nu_{u}^{w}(u)A_{u}^{(1)}}} d\left(\frac{A^{(1)}}{A^{(2)}}\right)_{u} \right)_{t}$$

- case of diffusive asset with (stochastic) vol $\sigma_{t}$, the vol target strategy converges to

$$e^{rt} \mathcal{E} \left( \int_{0}^{t} \sigma_{TV} \frac{\sigma_{u}}{\sqrt{\nu_{u}^{w}}} dW_{u} \right)_{t}$$

- the quality of the estimator is essential for tracking $\sigma_{TV}$
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How does it work in practice?

Source: Bloomberg 2014

- volatility visibly reduced, with trend following and smaller drawdowns
- what happens with realized volatility?
How does it work in practice?

- long-term window estimation yields vol around 10%
- short-term window shows vol-vol: the mechanism fails to deal with large moves

Source: Bloomberg 2014
Model dependence – basic tests

- fix \( w = 20 \) throughout the tests
- to check the approximation, look at 1Y ATM option on 10% target vol
- for simplicity, take underlying with 20% flat vol
  - replace the vol estimation with exact simulated vol – the implied ATM vol is exact
  - introducing vol estimator yields errors in excess of 1/2 vol point

fly in the ointment

- the volatility estimator is biased (Jensen’s inequality)
- Holtzman’s correction

\[
\tilde{\sigma} = C_N \sqrt{\tilde{\nu}}
\]

\[
C_N = \Gamma \left( \frac{N - 1}{2} \right) \sqrt{\frac{N - 1}{2}} / \Gamma \left( \frac{N}{2} \right)
\]
Model dependence – basic tests

- simplification, Brugger (1969): since \( \lim_{N \to \infty} \frac{N-1.5}{N-1} C_N^2 = 1 \), replace \( N - 1 \) by \( N - 1.5 \) in the estimation formula
- for simplicity, take underlying with 20% flat vol

![Graph showing 10% vol-target implied volatility](image)

- local vol results very similar, with extra 10bps errors
Model dependence – Jumps

Jump-diffusion model

\[
dS_t/S_t^- = \cdots - \lambda_t^S \, dt + \sqrt{v_t} \, dW_t + dN_t^S, \quad N_t^S := \sum_{i=1}^{N_t} (e^{Y_i} - 1)
\]

- Merton jumps
  - Poisson \( N_t \) with intensity \( \lambda \)
  - jump sizes are Gaussian, \( Y_i \sim N(\alpha, \delta^2) \)
- \( v \) can be any vol model, we focus on local vol

Calibration

- for a choice of jump parameters \( (\lambda, \alpha, \delta) \), local vol is calibrated
- calibration instruments: short-dated options, crash-cliquets
Model dependence — Jumps

► the “master formula” gives

\[
I(t) = e^{rt} \mathcal{E} \left( \int_0^t \sigma_{TV} \frac{\sigma_u}{\sqrt{\nu_u^w}} \, dW_u + \int_0^t \frac{\sigma_{TV}}{\sqrt{\nu_u^w}} \, dN_u^S \right)
\]  

(1)

► assume for simplicity \( \sigma_{\text{impl}} = 30\% \), take the Merton JD model calibrated to var swap prices

\[
\sigma^2 + \lambda(\alpha^2 + \delta^2) = \text{VarSwap1Y} = 30\%
\]

► assuming \( \sqrt{\nu_t^w} \approx \mathbb{E}[\sqrt{\nu_t^w}] = \sqrt{\text{VarSwap1Y}} \), we find

\[
I_t = e^{rt} \mathcal{E} \left( \int_0^t \frac{\sigma_{TV} \sigma}{\sqrt{\sigma^2 + \lambda(\alpha^2 + \delta^2)}} \, dW_u + \int_0^t \frac{\sigma_{TV}}{\sqrt{\sigma^2 + \lambda(\alpha^2 + \delta^2)}} \, dN_u^S \right)
\]  

\[
:= \sigma_I \quad \begin{matrix} \text{:=}\sigma_I \\ \text{:=}\gamma_I \end{matrix}
\]  

\]  

\[t\]
the 1Y var swap price of the index is precisely $\sigma_T^2$:

$$\text{VarSwap1Y}_i = \sigma_i^2 + \lambda \gamma_i^2 (\alpha^2 + \delta^2) = \sigma_T^2$$

so we’d expect little impact on vanillas, is this what we observe?

even with modest parameters $\lambda = 0.5$, $\alpha = -10.5\%$, $\delta = 4\%$ have significant differences
Model dependence – Jumps

- variance of the estimator plays an important role
- replacing $\sqrt{v_t^w}$ in the payoff with $\mathbb{E}[\sqrt{v_t^w}] = \sqrt{\text{VarSwap1Y}}$ gives much improved match
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Model dependence – Stochastic Volatility

LSV model

\[
dS_t / S_t = \cdots + \sqrt{v_t} \sigma(S_t, t) \, dW_t
\]

- Variance follows Heston or lognormal dynamics

Calibration

- For a choice of stochastic vol parameters \( \sigma(S, t) \) is calibrated to vanillas
- Forward-inductive PDE procedure

Pricing

- Again look at implied vol of \( I \), but put nontrivial cap in risky participation

\[
\alpha_n^{(1)} = \min(\text{cap}, \sigma_{TV} / \sqrt{v_{wn}})
\]
Model dependence – Stochastic Volatility

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10% vol-target implied volatility, 2Y implied vol, cap =\infty

10% vol-target implied volatility, 2Y implied vol, cap =150%

Strike

References
Model dependence – Stochastic Volatility

- cap of 150% is ineffective with $\sigma_{TV} = 10\%$, $\sigma_{impl} = 30\%$
- moving the cap further down, models produce different impact

![Graph showing 10% vol-target implied volatility, 2Y implied vol, cap =85%]
Conclusions

- Volatility estimator subtleties have impact on tracking target vol
- Any departure from pure lognormal case entails increase in tracking error
- Large impact of jumps
- Presence of cap has different impact depending on spot dynamics
- On an intuitive level
  - Cap introduces concavity in the participation function
    $$x \mapsto \min(\text{cap}, \sigma_{TV}/x)$$
  - This “concavity” brings down the expected value of participation in equity
  - The impact depends on the variance of the estimator, e.g. increase of vol-vol increases the impact
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Hedging

- vol control indices provide directional exposure with controlled risk
- buying options on those has become mainstream
- take example of vol control on SPX
Hedging

- look at 1Y ATM put on SPX5UT, struck at 21/08/2010
- during stable market conditions Delta hedging works fine
- as we (the hedger) are short $\Gamma$, large moves will hurt
- more than third of initial premium (2%) is lost during the crash
Hedging

- including vanilla options on the underlying can remove significant amount of jump risk, Xia and Sahakyan (2014)
- the hedging portfolio has additional component – Delta-hedged option on SPX
- the exact amount of SPX options can be determined using various criteria, say minimize the average portfolio move for jumps less than a threshold
A Regulatory Modelling challenge

- post-crisis regulatory-related modelling has significantly increased
- CVA, FVA, COLVA...
- typically, need to provide various statistics of the derivative portfolio in time under risk-neutral and physical measure
- eg. quantiles-based Potential Future Exposure (PFE) taken as a maximum amount of portfolio/trade exposure at a given confidence level
- assuming payoff $V_T$ at $T$, need $\Phi(X,t)$ s.t.

$$\Phi(X_t, t) := \mathbb{E}[V_T | \mathcal{F}_t](= \mathbb{E}[V_T | \mathcal{F}_t^X]), \quad 0 \leq t \leq T$$

- common approach American Monte Carlo (e.g. Sokol (2014)), entails
  - choosing state variables for each trade $X$
  - proper choice of basis functions for each state variable
  - estimation (regression) of $\Phi$
for simple trades entire state-space can be used for regression
- Asian option: spot and sum so far
for path-dependent, multiasset trades, situation less rosy
e.g. barrier option on 10-underlying basket/WO/BO with memory coupons
- full state-space prohibitively large
- rich structure of $\Phi(\cdot, t)$ due to presence of barriers, coupon payments etc.

...on the other hand, the value of those trades is (intuitively) driven by a few factors (basket/WO/BO + some trade specifics..)

we are in need of a consistent framework for
- choosing effective set of explanatory variables $\hat{X}$
- choice of the effective basis functions
- assessing the quality of approximation $\mathbb{E}[V_T | \mathcal{F}_t^{\hat{X}}] \approx \mathbb{E}[V_T | \mathcal{F}_t^X]$
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