2.1 Hilbert spaces, self-adjoint operators and the spectral theorem

In this section we briefly recall the theory of self-adjoint operators in Hilbert spaces. We will also use this opportunity to fix our notation.

2.1.1 Bounded operators

Let $\mathscr{V}$ be a complex vector space together with an inner product $(\cdot,\cdot)$, that is, a complex-valued function on $\mathscr{V}\times\mathscr{V}$ satisfying

$$(f,g)=\overline{(g,f)} \qquad \text{for all } f,g\in\mathscr{V},$$
$$(\alpha_1 f_1+\alpha_2 f_2,g)=\alpha_1(f_1,g)+\alpha_2(f_2,g) \qquad \text{for all } \alpha_1,\alpha_2\in\mathbb{C},\ f_1,f_2,g\in\mathscr{V},$$
$$(f,f)>0 \qquad \text{for all } 0\neq f\in\mathscr{V}.$$

Here and in all the following our sesqui-linear forms are linear in the first and anti-linear in the second argument.

If $(\cdot,\cdot)$ is an inner product on $\mathscr{V}$, then

$$\|f\|=\sqrt{(f,f)}$$

defines a norm on $\mathscr{V}$. The vector space $\mathscr{V}$, together with its inner product, is called a *Hilbert space* if it is complete with respect to this norm. It is called *separable* if it has a countable orthonormal basis.

From now on, let $\mathscr{H}$ be a separable Hilbert space.

We say that a sequence $(f_n)\subset\mathscr{H}$ *converges* to $f\in\mathscr{H}$ as $n\to\infty$ if $\|f_n-f\|\to 0$ as $n\to\infty$. We say that $(f_n)\subset\mathscr{H}$ *converges weakly* to $f\in\mathscr{H}$ as $n\to\infty$ if $(f_n,g)\to(f,g)$ for every $g\in\mathscr{H}$. If we want to emphasize the difference between convergence (in norm) and weak convergence we also call the former *strong convergence*.

The following facts about weak convergence are standard result from functional analysis. First, if $(f_n)\subset\mathscr{H}$ is weakly convergent, then $(\|f_n\|)$ is bounded. Second, the unit ball in $\mathscr{H}$ is weakly sequentially compact, that is, if $(\|f_n\|)$ is bounded, then there is a subsequence (f_{n_m}) which converges weakly to some $f\in\mathscr{H}$.

Next, we discuss operators on $\mathscr{H}$. A continuous, linear map $T:\mathscr{H}\to\mathscr{H}$ is called a *bounded* (linear) operator. This name comes from the fact that T is continuous if and only if

$$\|T\|=\sup_{\|f\|=1}\|Tf\|<\infty.$$

The set of bounded operators is complete with respect to the above norm.

By of the Riesz representation theorem for any bounded operator T on $\mathscr{H}$ one can define a unique bounded operator T^* on $\mathscr{H}$, called the *adjoint of* T, such that $(T^*f,g)=(f,Tg)$ for all $f,g\in\mathscr{H}$. This implies, in particular, that

$$\|T^*\| = \|T\|$$

and that, if $f_n \to f$ weakly in $\mathscr{H}$, then $Tf_n \to Tf$ weakly in $\mathscr{H}$.

An operator K on a Hilbert space $\mathscr{H}$ is called *compact* if the image of the closed unit ball in $\mathscr{H}$ is relatively compact. Clearly, compact operators are bounded and the product of a compact operator with a bounded operator is compact.

The next lemma connects the notion of compactness with that of weak convergence.

Lemma 2.1. *A bounded operator K is compact if and only if it transforms every weakly convergent sequence into a strongly convergent sequence.*

Proof. First, assume that K is compact and that $g_k \to 0$ weakly. Then, as recalled above, $Kg_k \to 0$ weakly. Moreover, as also mentioned before, $\sup\|g_k\| < \infty$. We argue by contradiction and assume that $\|Kg_k\| \not\to 0$. Thus, there is a subsequence (g_{k_l}) such that $\lim_{l\to\infty}\|Kg_{k_l}\| = \limsup_{k\to\infty}\|Kg_k\| =: a > 0$. Since (g_k) is bounded, the compactness of K implies that (Kg_k) is relatively compact and, therefore, there is a $h \in \mathscr{H}$ and a further subsequence $(g_{k_{l_m}})$ such that $Kg_{k_{l_m}} \to h$ strongly in $\mathscr{H}$ as $m \to \infty$. Since $Kg_k \to 0$ weakly, we have $h = 0$ and therefore $\|Kg_{k_{l_m}}\| \to 0$ as $m \to \infty$, contradicting $\|Kg_{k_l}\| \to a$ as $l \to \infty$.

To prove the converse implication, let (f_k) be a sequence with $\|f_k\| \leq 1$. Then, by the weak sequential compactness of the unit ball, there is a subsequence (f_{k_l}) which converges weakly. By assumption, (Kf_{k_l}) converges strongly. This shows that the image of the closed unit ball is relatively compact, as claimed. □

Lemma 2.2. *Let K be a compact operator. Then K^* is compact.*

Proof. Let (g_k) be a sequence which converges weakly to zero. According to Lemma 2.1 we need to prove that (K^*g_k) tends strongly to zero. We have

$$\|K^*g_k\|^2 = \langle KK^*g_k, g_k\rangle \leq \|KK^*g_k\|\|g_k\|\,. \tag{2.1}$$

By the uniform boundedness principle, since (g_k) converges weakly, the norms $\|g_k\|$ remain bounded. Moreover, as noted above, the boundedness of K^* implies that (K^*g_k) tends weakly to zero and therefore, by Lemma 2.1, (KK^*g_k) tends strongly to zero. Inserting this into (2.1) we infer that $\|K^*g_k\|$ tends to zero, as claimed. □

Let us conclude this subsection by discussing modes of convergence for operators. Let T_n and T be bounded operators on $\mathscr{H}$. We say that T is the *weak limit* of the T_n and write $T = \text{w-lim}_{n\to\infty}T_n$ if

$$(T_nf, g) \to (Tf, g) \qquad \text{for all } f, g \in \mathscr{H}\,.$$

We say that T is the *strong limit* of the T_n and write $T = \text{s-lim}_{n\to\infty}T_n$ if

$$\|(T_n - T)f\| \to 0 \qquad \text{for all } f \in \mathscr{H}\,.$$

Finally, we say that T is the *norm limit* of the T_n and write $T = \lim_{n\to\infty}T_n$ if

$$\|T_n - T\| \to 0\,.$$

Lemma 2.3. *Let K be a compact operator, let T,S be bounded operators and let $(T_n),(S_n)$ be sequences of bounded operators with $T = \mathrm{s}-\lim_{n\to\infty} T_n$ and $S = \mathrm{s}-\lim_{n\to\infty} S_n$. Then $\lim_{n\to\infty} T_n K S_n^* = TKS^*$ (in norm).*

Proof. Step 1. We first prove the assertion in the case where $S_n = S$ for all n and we abbreviate $L = KS^*$. Assume the stated convergence would not hold. Then there are $\varepsilon > 0$ and, passing to a subsequence if necessary, f_n such that $\|f_n\| = 1$ and $\|(T_n - T)Lf_n\| \geq \varepsilon$. By weak compactness, passing to another subsequence if necessary, we may assume that $f_n \rightharpoonup f$ for some f. Since L is compact as the product of a bounded and a compact operator, we infer from Lemma 2.1 that $Lf_n \to Lf$ in norm. Writing

$$\begin{aligned}\|(T_n - T)Lf_n\| &\leq \|(T_n - T)Lf\| + \|(T_n - T)L(f_n - f)\| \\ &\leq \|(T_n - T)Lf\| + (\|T_n\| + \|T\|)\|Lf_n - Lf\|\,,\end{aligned}$$

we see that the first term on the right side tends to zero by strong convergence of T_n and the second term tends to zero since $\|T_n\|$ remains bounded by the uniform boundedness principle `Reference!`. Thus, $\|(T_n - T)Lf_n\| \to 0$, a contradiction.

Step 2. We now prove the assertion in the case where $T_n = T$ for all n and we abbreviate $M = TK$. Since M is compact as the product of a bounded and a compact operator, we infer from Lemma 2.2 that M^* is compact. Therefore, Step 1 implies that $S_n M^* \to SM^*$ in operator norm. Since the norm of the adjoint equals the norm of the operator itself, this implies that $MS_n^* \to MS^*$ in operator norm, which proves the assertion in th.

Step 3. To prove the assertion in the general case, we write

$$T_n K S_n^* - TKS^* = (T_n - T)KS_n^* + TK(S_n^* - S^*)\,.$$

The first term on the right side tends to zero in operator norm by Step 1 (applied with S replaced by the identity) and the fact that $\|S_n^*\| = \|S_n\|$ is uniformly bounded, and the second term tends to zero in operator norm by Step 2. This concludes the proof of the proposition. □

2.1.2 Unbounded operators

An (unbounded, linear) operator in $\mathscr{H}$ is a linear map T from its domain $\operatorname{dom} T$, a linear subspace of $\mathscr{H}$, into $\mathscr{H}$. (We emphasize that for us, a linear subspace of $\mathscr{H}$ is a linear subset of $\mathscr{H}$, without assuming it is closed in $\mathscr{H}$. This notion is sometimes used differently by other authors.) In particular, two operators T and S coincide if $\operatorname{dom} S = \operatorname{dom} T$ and $Tf = Sf$ for all $f \in \operatorname{dom} S = \operatorname{dom} T$.

The operator T is called *closed* if $\operatorname{dom} T$ is complete with respect to the norm $(\|Tf\|^2 + \|f\|^2)^{1/2}$. Clearly, every bounded operator defined on a closed subspace of $\mathscr{H}$ is closed. Note, that if a closed operator T is invertible, then its inverse T^{-1} is closed as well.

The operator T is called *densely defined* if $\operatorname{dom} T$ is dense in $\mathscr{H}$. For such a T we now define the *adjoint* T^* as follows,

$$\begin{aligned}\operatorname{dom} T^* = \{f \in \mathscr{H} :\ & \text{there is a } g \in \mathscr{H} \text{ such that} \\ & \text{for all } h \in \operatorname{dom} T \text{ one has } (g,h) = (f,Th)\} \,.\end{aligned}$$

Since T is densely defined, for $f \in \operatorname{dom} T^*$ the corresponding g is unique and we define $T^* f = g$. For bounded T, this coincides with the definition given above.

One can show that T^* is always closed. Moreover, if T is closed, then T^* is densely defined and $T^{**} = (T^*)^* = T$. Moreover, note that

$$\ker T^* = (\operatorname{ran} T)^\perp \,, \tag{2.2}$$

which, in turn, implies that

$$(\ker T^*)^\perp = \overline{\operatorname{ran} T} \,. \tag{2.3}$$

For a closed operator T, the *resolvent set* $\rho(T)$ is defined by

$$\begin{aligned}\rho(T) = \{\lambda \in \mathbb{C} :\ & T - \lambda \text{ is a bijection from } \operatorname{dom} T \text{ onto } \mathscr{H} \\ & \text{with a bounded inverse}\}\end{aligned}$$

and the operator $(T-\lambda)^{-1}$ is called the *resolvent* of T at $\lambda \in \rho(T)$. Using the closed graph theorem one could deduce the boundedness of the inverse from the fact that the range of $T - \lambda$ is equal to $\mathscr{H}$, although we will not use this fact explicitly in this book.

The *spectrum* $\sigma(T)$ is defined by

$$\sigma(T) = \mathbb{C} \setminus \rho(T)$$

and the *point spectrum* is defined by

$$\sigma_p(T) = \{\lambda \in \sigma(T) :\ \ker(T - \lambda) \neq \{0\}\} \,.$$

An number $\lambda \in \sigma_p(T)$ is called an *eigenvalue* of T and $0 \neq f \in \ker(T-\lambda)$ is called an *eigenvector* of T.

2.1.3 Self-adjoint operators and spectrum

A densely defined operator A is called *symmetric* if $\operatorname{dom} A \subset \operatorname{dom} A^*$ and $A^* f = Af$ for every $f \in \operatorname{dom} A$. Equivalently, A is symmetric if and only if

$$(Af,g) = (f,Ag) \qquad \text{for all } f,g \in \operatorname{dom} A \,.$$

This in turn is equivalent to

$$(Af,f) \in \mathbb{R} \qquad \text{for all } f \in \operatorname{dom} A\,.$$

Note that eigenvalues of a symmetric operator are necessarily real, since $Af = \lambda f$ implies

$$\lambda \|f\|^2 = (Af,f) = (f,Af) = \overline{\lambda}\|f\|^2\,.$$

Moreover, if f and g are eigenvectors of a symmetric operator A corresponding to different eigenvalues μ and λ, then $(f,g) = 0$. Indeed,

$$\lambda(f,g) = (Af,g) = (f,Ag) = \overline{\mu}(f,g) = \mu(f,g)\,.$$

An operator A is called *self-adjoint* if it is densely defined, symmetric and $\operatorname{dom} A = \operatorname{dom} A^*$. Clearly, any self-adjoint operator is closed.

Lemma 2.4. *Let A be a self-adjoint operator. Then we have $\sigma(A) \subset \mathbb{R}$. Moreover, $\lambda \in \rho(A)$ if and only if there is an $\varepsilon > 0$ such that*

$$\|(A-\lambda)f\| \geq \varepsilon\|f\|\,, \quad f \in \operatorname{dom} A\,. \tag{2.4}$$

Proof. Indeed, for any symmetric operator A,

$$\|(A-\lambda)f\|^2 = \|(A-\operatorname{Re}\lambda)f\|^2 + (\operatorname{Im}\lambda)^2\|f\|^2 \geq (\operatorname{Im}\lambda)^2\|f\|^2\,.$$

Hence, for any $\operatorname{Re}\lambda \neq 0$ the operator $A-\lambda$ is injective and its inverse, defined on $\operatorname{ran}(A-\lambda)$ is bounded. If, in addition, A is closed the inequality above implies that $\operatorname{ran}(A-\lambda)$ is closed. Hence, if A is self-adjoint, by (2.3) for $\operatorname{Re}\lambda \neq 0$ we get

$$\operatorname{ran}(A-\lambda) = \overline{\operatorname{ran}(A-\lambda)} = (\ker(A^*-\overline{\lambda}))^\perp = (\ker(A-\overline{\lambda}))^\perp = \mathscr{H}\,,$$

and $\lambda \in \rho(A)$. Thus $\sigma(A) \subset \mathbb{R}$, as claimed. The same argument gives that for all $\lambda \in \mathbb{R}$ for which (2.4) holds one has $\lambda \in \rho(A)$.

Conversely, if $\lambda \in \sigma_p(A)$, then (2.4) clearly fails for eigenvectors f. If $\lambda \in \sigma(A) \setminus \sigma_p(A) \subset \mathbb{R}$, then $A-\lambda$ is invertible. We show that this inverse is unbounded which contradicts (2.4). Indeed, as above we have $\overline{\operatorname{ran}(A-\lambda)} = (\ker(A-\lambda))^\perp = \mathscr{H}$. Since A is closed, the inverse of $A-\lambda$ is closed. For a bounded inverse of $A-\lambda$ this would mean that its domain $\operatorname{ran}(A-\lambda)$ is closed in $\mathscr{H}$ and $\operatorname{ran}(A-\lambda) = \mathscr{H}$. But then $\lambda \in \rho(A)$ which contradicts $\lambda \in \sigma(A)$. □

2.1.4 The spectrum of a multiplication operator

Let X be a set, $\mathscr{A}$ a sigma-algebra on X and μ a non-negative measure on $(X,\mathscr{A})$. We recall that the measure space $(X,\mathscr{A},\mu)$ is called *separable* if there is a countable subset $\mathscr{B} \subset \mathscr{A}$ such that for any $E \in \mathscr{A}$ with $\mu(E) < \infty$ and any $\varepsilon > 0$ there is a $B \in \mathscr{B}$ with $\mu(B\Delta E) \leq \varepsilon$. In that case, the Hilbert space $L^2(X,\mathscr{A},\mu)$ is separable.

We also assume that $(X,\mathscr{A},\mu)$ is sigma-finite.

Let φ be a measurable, complex-valued function on X, which is finite almost everywhere, and consider the multiplication operator T_φ in $L^2(X)$ defined by

$$T_\varphi f = \varphi f\,, \qquad \operatorname{dom} T_\varphi = \left\{ f \in L^2(X) :\ \varphi f \in L^2(X) \right\}.$$

The completeness of $L^2(X,(1+|\varphi|^2)\mu)$ implies that T_φ is a closed operator. Moreover, T_φ is densely defined, because for any $f\in L^2(X)$ the functions $\chi_{\{|\varphi|\leq n\}} f \in \operatorname{dom} T_\varphi$ converge to f as $n\to\infty$ by dominated convergence since φ is finite almost everywhere. The adjoint of T_φ is given by

$$T_\varphi^* = T_{\overline{\varphi}}\,.$$

In particular, T_φ is self-adjoint if and only if φ is real-valued almost everywhere.

Note that T_φ is bounded if and only if φ is essentially bounded, and in this case

$$\|T_\varphi\| = \|\varphi\|_{L^\infty}\,.$$

Let us characterize the spectrum of the operator T_φ. We claim that

$$\sigma_p(T_\varphi) = \{z\in\mathbb{C} :\ \mu(\varphi^{-1}(\{z\}))>0\}\,,$$

where, for any $E\subset\mathbb{C}$, $\varphi^{-1}(E)=\{x\in X :\ \varphi(x)\in E\}$ is the pre-image of E under φ. Indeed, if $\varphi f = zf$ almost everywhere on X for some $f\not\equiv 0$, then $\varphi = z$ almost everywhere on the set $\{x\in X :\ f(x)\neq 0\}$, which has positive measure. Conversely, if $Y\subset\varphi^{-1}(\{z\})$ is measurable with $0<\mu(Y)<\infty$, then $0\not\equiv\chi_Y\in\operatorname{dom}T_\varphi$ and $T_\varphi\chi_Y = z\chi_Y$.

Next, we claim that

$$\sigma(T_\varphi) = \{z\in\mathbb{C} :\ \mu(\varphi^{-1}(\{\zeta\in\mathbb{C} :\ |\zeta-z|\leq\varepsilon\}))>0 \text{ for all } \varepsilon>0\}\,, \qquad (2.5)$$

that is, $\sigma(T_\varphi)$ is the essential range of φ. Let us prove this. Clearly, if $z\in\sigma_p(T_\varphi)$, then z belongs to the set on the right side. If $z\notin\sigma_p(T_\varphi)$, then the operator $T_\varphi - z$ is invertible on $\operatorname{ran} T_\varphi$ and the inverse is given by T_{ψ_z}, where

$$\psi_z(x) = \frac{1}{\varphi(x)-z} \qquad \text{for all } x\in X\,.$$

As we have noticed before, this operator is bounded if and only if ψ_z is essentially bounded, that is, if and only if there is an $\varepsilon>0$ such that $|\varphi(x)-z|\geq\varepsilon$ for almost every $x\in X$. This means that z does not belong to the right side in (2.5). This proves (2.5).

2.1.5 The spectral theorem

A bounded operator P is called an *orthogonal projection* if $P = P^* = P^2$. A *projection valued measure* is a map $P : \omega \mapsto P_\omega$ on the Borel sigma-algebra on $\mathbb{R}$ taking values in the set of orthogonal projections such that

(a) $P_\emptyset = 0$, $P_\mathbb{R} = \mathbb{I}$,
(b) If $\omega = \bigcup_{j\in\mathbb{N}} \omega_j$ with $\omega_n \cap \omega_m = \emptyset$ for $n \neq m$, then $P_\omega = s - \lim_{N\to\infty} \sum_{n=1}^N P_{\omega_n}$.
(c) $P_{\omega_1} P_{\omega_2} = P_{\omega_1 \cap \omega_2}$ for all Borel sets ω_1, ω_2.

Now given a projection valued measure, we shall define a corresponding self-adjoint operator A. For each $f \in \mathscr{H}$, $\omega \mapsto (P_\omega f, f)$ is a non-negative Borel measure on $\mathbb{R}$ which we denote by $d(P_\lambda f, f)$. Since $P_\mathbb{R} = \mathbb{I}$, we have

$$\int_\mathbb{R} d(P_\lambda f, f) = \|f\|^2 .$$

The complex measure $d(P_\lambda f, g)$ is defined by polarization, namely

$$\begin{aligned} d(P_\lambda f, g) = \frac{1}{4} &(d(P_\lambda (f+g), f+g) + id(P_\lambda (f+ig), f+ig) \\ &-d(P_\lambda (f-g), f-g) - id(P_\lambda (f-ig), f-ig)) . \end{aligned}$$

It follows from the defining properties of a projection valued measure that the set

$$\operatorname{dom} A = \left\{ f \in \mathscr{H} : \int_\mathbb{R} \lambda^2 \, d(P_\lambda f, f) < \infty \right\}$$

is dense in $\mathscr{H}$. For fixed $f \in \operatorname{dom} A$ the mapping $g \mapsto \int_\mathbb{R} \lambda \, d(P_\lambda f, g)$ is anti-linear and, since

$$\left| \int_\mathbb{R} \lambda d(P_\lambda f, g) \right|^2 \leq \int_\mathbb{R} \lambda^2 d(P_\lambda f, f) \int_\mathbb{R} d(P_\lambda g, g) = \int_\mathbb{R} \lambda^2 \, d(P_\lambda f, f) \, \|g\| ,$$

it is bounded. Therefore, by Riesz' representation theorem, there is a unique $Af \in \mathscr{H}$ such that

$$(Af, g) = \int_\mathbb{R} \lambda d(P_\lambda f, g) \qquad \text{for all } g \in \mathscr{H} .$$

This defines an operator A and we write

$$A = \int_\mathbb{R} \lambda \, dP_\lambda .$$

The operator A is self-adjoint. In fact, its symmetry follows easily from the self-adjointness of P_ω. For the self-adjointness, let $g \in \operatorname{dom} A^*$ and, for $\Lambda > 0$, choose $f = AP_{[-\Lambda,\Lambda]} g \in \operatorname{dom} A$. Then

$$\int_{[-\Lambda,\Lambda]} \lambda^2 \, d(P_\lambda g, g) = (g, Af) = (A^* g, f) \le \|A^* g\| \|f\| = \|A^* g\| \sqrt{\int_{[-\Lambda,\Lambda]} \lambda^2 \, d(P_\lambda g, g)}$$

Thus,

$$\int_{[-\Lambda,\Lambda]} \lambda^2 \, d(P_\lambda g, g) \le \|A^* g\|^2 .$$

As $\Lambda \to \infty$, monotone convergence implies that $g \in \operatorname{dom} A$, which proves the A is self-adjoint.

The content of the spectral theorem is that this procedure can be inverted.

Theorem 2.5. *Let A be a self-adjoint operator in $\mathscr{H}$. Then there is a unique projection valued measure P on $\mathscr{H}$ such that*

$$A = \int_{\mathbb{R}} \lambda \, dP_\lambda .$$

In the situation of the theorem the projection valued measure is also called the *spectral measure* of A. It allows one to define functions of A much in the same way as we defined before an operator corresponding to a projection valued measure. More precisely, if φ is a complex-valued Borel function on $\mathbb{R}$, then

$$\operatorname{dom} \varphi(A) = \left\{ f \in \mathscr{H} : \int_{\mathbb{R}} |\varphi(\lambda)|^2 d(P_\lambda f, f) < \infty \right\}$$

and

$$(\varphi(A) f, g) = \int_{\mathbb{R}} \varphi(\lambda) \, d(P_\lambda f, g) \qquad \text{for all } g \in \mathscr{H} .$$

This defines a functional calculus: If φ is a measurable functions on $\mathbb{R}$, then

$$\varphi(A)^* = \overline{\varphi}(A) , \tag{2.6}$$

$$1(A) = \mathbb{I} , \tag{2.7}$$

$$\|\varphi(A) f\|^2 = \int_{\mathbb{R}} |\varphi(\lambda)|^2 d(P_\lambda f, f) \quad \text{for all} \quad f \in \operatorname{dom} \varphi(A) , \tag{2.8}$$

$$\|\varphi(A)\| = P\text{-}\sup_\lambda |\varphi(\lambda)| . \tag{2.9}$$

Here, for a measurable function χ on $\mathbb{R}$,

$$P\text{-}\sup_\lambda \chi(\lambda) = \inf \left\{ a \in \mathbb{R} : P_{\{\chi > a\}} = 0 \right\} .$$

If the functions φ and ψ are bounded, or more generally if they satisfy P-a.e. the conditions $|\varphi| + |\psi| \le C(1 + |\alpha\varphi + \beta\psi|)$ for some $\alpha, \beta \in \mathbb{C}$, or $|\varphi| + |\psi| \le C(1 + |\varphi\psi|)$ respectively, then

$$\alpha\varphi(A) + \beta\psi(A) = (\alpha\varphi + \beta\psi)(A) , \tag{2.10}$$

$$\varphi(A)\psi(A) = (\psi\varphi)(A) , \quad \psi(A)\varphi(A) = (\psi\varphi)(A) . \tag{2.11}$$

In particular, the operators $\varphi(A)$ for $\varphi(\lambda)=\lambda^n$, $n\in\mathbb{N}$, coincide with n-fold product of A defined in the sense of operator products.

In the case of general unbounded functions φ and ψ some care is needed, since sums or products of operator functions might not be closed. The corresponding identities hold pointwise on the domains of the operator expressions on the left side, which are dense in the operator norm of the right hand sides.

We point out, that this functional calculus also a continuity property. If $\varphi_n\to\varphi$ pointwise and $|\varphi_n(\lambda)|\le C(1+|\varphi(\lambda)|)$ for all $\lambda\in\mathbb{R}$ and all n, then $\varphi_n(A)f\to\varphi(A)f$ for all $f\in\operatorname{dom}\varphi(A)$. This fact is used to prove the properties above in the unbounded case, but it is also of general importance.

The spectral projection P_M of some Borel-measurable set M reappears as the characteristic function χ_M of A, namely one has $P_M=\chi_M(A)$. In particular, for $f\in\operatorname{dom}\varphi(A)\cap P_M\mathscr{H}$ because of (2.8) it holds

$$\|\varphi(A)f\|^2=\|\varphi(A)\chi_M(A)f\|^2=\int_M|\varphi(\lambda)|^2d(P_\lambda f,f)\,. \tag{2.12}$$

Let

$$\operatorname{supp}P=\{\lambda\in\mathbb{R}:\ \text{for all }\varepsilon>0\ P_{(\lambda-\varepsilon,\lambda+\varepsilon)}\neq 0\}$$

be the support of the spectral measure P. Based on (2.12) one can show that

Lemma 2.6. $\sigma(A)=\operatorname{supp}P\,.$

Proof. Assume first that $\lambda\notin\operatorname{supp}P$. Then, for some $\varepsilon>0$ we have $P_{(\lambda-\varepsilon,\lambda+\varepsilon)}=0$. Hence, by (2.12) with $\varphi(\mu)=\mu-\lambda$ and $M=(-\infty,\lambda-\varepsilon]\cup[\lambda+\varepsilon,+\infty)$ we get

$$\|(A-\lambda)f\|^2=\int_M|\mu-\lambda|^2d(P_\mu f,f)\ge\varepsilon^2\int_M d(P_\mu f,f)=\varepsilon^2\int d(P_\mu f,f)=\varepsilon^2\|f\|^2\,.$$

Hence, by (2.4) we have $\lambda\in\rho(A)$. Conversely, if $\lambda\in\operatorname{supp}P$, then for any $n\in\mathbb{N}$ there is a $f_n\neq 0$ with $f_n\in P_{(\lambda-1/n,\lambda+1/n)}\mathscr{H}$. From (2.12) with $M'=(\lambda-1/n,\lambda+1/n)$ we see, that

$$\|(A-\lambda)f_n\|^2=\int_{M'}|\mu-\lambda|^2d(P_\mu f_n,f_n)\le\frac{1}{n^2}\int_{M'}d(P_\mu f_n,f_n)=\frac{1}{n^2}\|f_n\|^2\,.$$

Therefore the bound (2.4) is violated and $\lambda\in\sigma(A)$.

Definition 2.7. An operator A with domain $\operatorname{dom}A$ is called *lower semibounded* in a Hilbert space $\mathscr{H}$ if it is symmetric and

$$m_A=\inf_{0\neq u\in\operatorname{dom}A}\frac{(Au,u)}{\|u\|^2}>-\infty\,.$$

Lemma 2.8. *Let A be a self-adjoint, lower semibounded operator. Then*

$$m_A=\inf\sigma(A)\,.$$

Moreover, m_A is an eigenvalue of A if and only if the infimum $\inf_{0\neq u\in \operatorname{dom} A}(Au,u)/\|u\|^2$ *is a minimum, and eigenvectors are precisely those vectors for which the infimum is attained.*

Proof. First note that for any $\lambda < m_A$ and $u \in \operatorname{dom} A$ we have

$$(m_A - \lambda)\|u\|^2 \leq (Au,u) - \lambda\|u\|^2 = ((A-\lambda)u,u) \leq \|(A-\lambda)u\|\|u\|,$$

and so, by Lemma 2.4, the point λ is in the resolvent set. Hence, $\sigma(A) \subset [m_A, +\infty)$ and, in particular, $m_A \leq \inf\sigma(A)$. Conversely, by the spectral theorem and Lemma 2.6,

$$(Au,u) = \int_{\mathbb{R}} \lambda\, d(P_\lambda u,u) = \int_{\sigma(A)} \lambda\, d(P_\lambda u,u) \geq \inf\sigma(A)\|u\|^2 \qquad \text{for all } u \in \operatorname{dom} A.$$

Since, by definition, m_A is the optimal lower bound, we conclude that $m_A \geq \inf\sigma(A)$. Hence, $m_A = \inf\sigma(A)$.

Finally, equality in the previous bound is attained if and only if $d(P_\lambda u,u)$ is a point measure of mass $\|u\|^2$ supported at $\lambda = m_A$. The latter is equivalent to the fact that $u = P(\{m_A\})u$, which in turn is equivalent to u being an eigenvector of A with eigenvalue m_A. □

Let us now compare the original spectral measure $P = P(A)$ of A and the spectral measure $P(\varphi(A))$ of $\varphi(A)$, where φ is a measurable, real-valued function. Then the spectral mapping theorem holds true, namely

$$P_\omega(\varphi(A)) = P_{\varphi^{-1}(\omega)}(A) \tag{2.13}$$

for all Borel sets $\omega \subset \mathbb{R}$. In particular, this implies

$$\sigma(\varphi(A)) = \overline{\varphi(\sigma(A))}$$

for all functions φ, which are also continuous on $\operatorname{supp} P$.

2.1.6 The essential spectrum and Weyl's theorem

Let A be a self-adjoint operator in a Hilbert space $\mathscr{H}$ with spectral measure P. Let us define the *essential spectrum* $\sigma_{\text{ess}}(A)$ and the *discrete spectrum* $\sigma_{\text{disc}}(A)$ of A as follows,

$$\sigma_{\text{ess}}(A) = \{\lambda \in \mathbb{R} : \dim P_{(\lambda-\varepsilon,\lambda+\varepsilon)}\mathscr{H} = \infty \text{ for all } \varepsilon > 0\},$$
$$\sigma_{\text{disc}}(A) = \sigma(A) \setminus \sigma_{\text{ess}}(A) = \{\lambda \in \sigma(A) : \dim P_{(\lambda-\varepsilon,\lambda+\varepsilon)}\mathscr{H} < \infty \text{ for some } \varepsilon > 0\}.$$

It is easy to prove that $\lambda \in \sigma_{\text{disc}}(A)$ if and only if λ is an isolated point of $\sigma(A)$ (i.e., $\sigma(A) \cap (\lambda-\varepsilon,\lambda+\varepsilon) = \{\lambda\}$ for some $\varepsilon > 0$) and λ is an eigenvalue of finite multiplicity. Moreover, one can prove that $\lambda \in \sigma_{\mathrm{ess}}(A)$ if and only if one or more of the following holds: $\mathrm{ran}(A-\lambda)$ is not closed, or λ is an accumulation point of eigenvalues, or λ is an eigenvalue of infinite multiplicity.

In practice it is useful to have a criterion of whether a point belongs to the essential spectrum of A which is not expressed through the spectral measure of A, but through A itself. To give this criterion we shall say that a sequence $(u_n) \subset \mathscr{H}$ is a *singular sequence* or *Weyl sequence* for A at a point $\lambda \in \mathbb{R}$ if the following conditions are satisfies:

$$\inf_n \|u_n\| > 0\,, \tag{2.14}$$

$$u_n \to 0 \text{ weakly in } \mathscr{H}\,, \tag{2.15}$$

$$u_n \in \mathrm{dom} A\,, \tag{2.16}$$

$$(A-\lambda)u_n \to 0 \text{ strongly in } \mathscr{H}\,. \tag{2.17}$$

Lemma 2.9. *A point $\lambda \in \mathbb{R}$ belongs to $\sigma_{\mathrm{ess}}(A)$ if and only if there is a singular sequence for A at λ.*

Proof. First, let $\lambda \in \sigma_{\mathrm{ess}}(A)$ and let (ε_n) be a decreasing sequence of positive numbers tending to zero. Then, by the definition of the essential spectrum, there is an orthonormal system (u_n) with $u_n \in P_{(\lambda-\varepsilon_n,\lambda+\varepsilon_n)}\mathscr{H}$ for all n. Then (2.14), (2.15) and (2.16) are clearly satisfied and (2.17) follows from

$$\begin{aligned}\|(A-\lambda)u_n\|^2 &= \int_{(\lambda-\varepsilon_n,\lambda+\varepsilon_n)} (t-\lambda)^2 d(P_t u_n, u_n)\\ &\le \varepsilon_n^2 \int_{(\lambda-\varepsilon_n,\lambda+\varepsilon_n)} d(P_t u_n, u_n) = \varepsilon_n^2 \to 0\,.\end{aligned}$$

Conversely, assume that there is a singular sequence (u_n) for A at λ. We argue by contradiction, assuming that $\lambda \notin \sigma_{\mathrm{ess}}(A)$. Then $\dim P_{(\lambda-\varepsilon,\lambda+\varepsilon)}\mathscr{H} < \infty$ for some $\varepsilon > 0$. By decreasing ε if necessary (but keeping it positive) we can assure that

$$\sigma(A) \cap (\lambda-\varepsilon, \lambda+\varepsilon) \subset \{\lambda\}\,.$$

Let $v_n := u_n - P_{\{\lambda\}}u_n$. Then $v_n \in \left(P_{\{\lambda\}}\mathscr{H}\right)^{\perp}$ and, by (2.17), $(A-\lambda)v_n = (A-\lambda)u_n \to 0$ in $\mathscr{H}$. By the choice of ε, we have $\|(A-\lambda)v_n\| \ge \varepsilon\|v_n\|$ and therefore $v_n \to 0$ in $\mathscr{H}$. Since $P_{\{\lambda\}}$ has finite rank, it is compact and therefore (2.15) implies that $P_{\{\lambda\}}u_n \to 0$ in $\mathscr{H}$. Thus, also $u_n = v_n + P_{\{\lambda\}}u_n \to 0$ in $\mathscr{H}$. This contradicts (2.14) and completes the proof. □

The following theorem is due to Weyl and states the stability of the essential spectrum under certain perturbations. This is very useful in practice since it reduces the computation of the essential spectrum to that for certain model operators. This is an example of a perturbation theoretic result.

Theorem 2.10. *Let A_1 and A_2 be self-adjoint operators and assume that for some $z \in \rho(A_1) \cap \rho(A_2)$,*

$$(A_1 - z)^{-1} - (A_2 - z)^{-1} \qquad \textit{is compact}\,. \tag{2.18}$$

Then $\sigma_{\rm ess}(A_1) = \sigma_{\rm ess}(A_2)$.

It follows from the resolvent identity

$$(A_j - z)^{-1} - (A_j - z')^{-1} = (z - z')(A_j - z)^{-1}(A_j - z')^{-1}\,, \quad z, z' \in \rho(A_j)\,, \quad j = 1, 2\,,$$

that (2.18) holds for some $z \in \rho(A_1) \cap \rho(A_2)$ if and only if it holds for any $z \in \rho(A_1) \cap \rho(A_2)$.

Clearly, because of

$$(A_1 - z)^{-1} - (A_2 - z)^{-1} = -(A_1 - z)^{-1}(A_1 - A_2)(A_2 - z)^{-1}\,, \quad z \in \rho(A_1) \cap \rho(A_2)\,,$$

the assumption (2.18) holds if A_1 and A_2 are bounded operators with $A_1 - A_2$ compact. In our applications it is crucial, however, that it suffices that the compactness holds only for the resolvent difference.

Proof. Since the assertion is symmetric in the operators A_1 and A_2, it suffices to prove that $\sigma_{\rm ess}(A_1) \subset \sigma_{\rm ess}(A_2)$. Let $\lambda \in \sigma_{\rm ess}(A_1)$. Then, by Lemma 2.9, there is a singular sequence (u_n) for A_1 at λ. With z from (2.18) let $v_n = (A_2 - z)^{-1}(A_1 - z)u_n$ (which is well-defined because of (2.16)). We would like to show that (v_n) is a singular sequence for A_2 at λ. Once this is done, the theorem follows again from Lemma 2.9.

Obviously, (2.16) is satisfied for v_n and A_2.

Let us verify (2.14) and (2.15). Denoting $K = (A_2 - z)^{-1} - (A_1 - z)^{-1}$, we have

$$v_n = K(A_1 - z)u_n + u_n = K(A_1 - \lambda)u_n + (\lambda - z)Ku_n + u_n\,.$$

Because of (2.15) and (2.17) for u_n and A_1 and Lemma 2.1 we have $v_n - u_n \to 0$ in $\mathscr{H}$. Therefore (2.14) and (2.15) for u_n and A_1 imply (2.14) and (2.15) for v_n and A_2.

Finally, in order to verify (2.17), we compute

$$(A_2 - \lambda)v_n = (A_1 - z)u_n + (z - \lambda)v_n = (A_1 - \lambda)u_n + (z - \lambda)(v_n - u_n)\,.$$

By (2.17) for u_n and A_1 and by the fact that $v_n - u_n \to 0$ in $\mathscr{H}$, we obtain (2.17) for v_n and A_2. This completes the proof. □

In Subsection 2.2.6, after having recalled the concept of a lower semibounded quadratic form, we will state a quadratic form version of Weyl's theorem.

Here is a consequence of the spectral theorem for compact operators.

Lemma 2.11. *Let T be self-adjoint compact operator in $\mathscr{H}$ with* $\dim \mathscr{H} = \infty$. *Then* $\sigma_{\rm ess}(T) = \{0\}$ *and $\mathscr{H}$ has an orthonormal basis consisting of eigenfunctions of T.*

Proof. By the resolvent identity, the assumption (2.18) in Weyl's theorem holds with $A_1 = T$ and $A_2 = 0$. Thus, the theorem implies that $\sigma_{\rm ess}(T) = \sigma_{\rm ess}(0) = \{0\}$. This means that the spectrum of T away from zero consists of isolated eigenvalues of finite multiplicities. Therefore, there are either only finitely many non-zero eigenvalues, or there is an infinite number of them and they accumulate at zero. The spectral theorem then provides an orthonormal basis of eigenfunctions of T. □

Lemma 2.11 can be supplemented by the reverse statement.

Lemma 2.12. *Let T be a bounded self-adjoint operator in $\mathscr{H}$ with* $\dim H = \infty$. *If* $\sigma_{\mathrm{ess}}(T) = \{0\}$*, then T is compact.*

Proof. We apply Lemma 2.1. Assume that $u_n \to 0$ weakly in $\mathscr{H}$. Let $\tau > 0$ and $P = \chi_{(-\tau,\tau)}(T)$. Since $\sigma_{\mathrm{ess}}(T) = \{0\}$, the operator $P^{\perp} = \mathbb{I} - P$ has finite rank. Therefore the weak convergence $u_n \to 0$ implies the strong convergence $P^{\perp}Tu_n \to 0$. On the other hand, by the spectral theorem, $\|PTu_n\| \leq \tau\|u_n\|$. Since by orthogonality $\|Tu_n\|^2 = \|P^{\perp}Tu_n\|^2 + \|PTu_n\|^2$, we find that $\limsup_n \|Tu_n\|^2 \leq \tau^2 C$ with $C = \sup_n \|u_n\|^2$. Note that by uniform boundedness C is finite. Since $\tau > 0$ can be chosen arbitrarily small, we conclude that $Tu_n \to 0$ strongly. □

2.2 Semibounded operators and forms and the variational principle

2.2.1 Semibounded operators and forms

In this section we consider quadratic forms a with domain $d[a]$ which is always assumed to be dense.

Definition 2.13. A quadratic form a with domain $d[a]$ is called *lower semibounded* in a Hilbert space $\mathscr{H}$ if it is real-valued and

$$m_a = \inf_{0\neq u\in d[a]} \frac{a[u]}{\|u\|^2} > -\infty\,.$$

Let $a[\cdot,\cdot]$ be the sesquilinear form obtained from a by polarization. Then for each $m > -m_a$, the expression

$$a[u,v] + m(u,v)$$

defines a scalar product on $d[a]$ and for different $m > -m_a$ the corresponding norms are equivalent.

Definition 2.14. A lower semibounded quadratic form a with domain $d[a]$ is called *closed* in a Hilbert space $\mathscr{H}$ if for some (and hence any) $m > -m_a$ the set $d[a]$ is complete with respect to the norm

$$\left(a[u] + m\|u\|^2\right)^{1/2}.$$

A subspace $F \subset d[a]$, which is dense with respect to the norm $\left(a[u] + m\|u\|^2\right)^{1/2}$ in $d[a]$, is called a *form core*.

We now discuss the relation between quadratic forms and operators. We recall the notion of a lower semibounded operator A from Definition 2.7 and the notation

$$m_A = \inf_{0\neq u\in \operatorname{dom} A} \frac{(Au,u)}{\|u\|^2} > -\infty\,.$$

Now let A be a self-adjoint, lower semibounded operator. We define a quadratic form

$$a[u] = ((A-m_A)^{1/2}u, (A-m_A)^{1/2}u) + m_A\|u\|^2 \tag{2.19}$$

with domain

$$d[a] = \operatorname{dom}(A-m_A)^{1/2}\,.$$

Note that by Lemma 2.8 we have $\inf\sigma(A-m_A) = 0$ and therefore the square root $(A-m_A)^{1/2}$ is well-defined as a self-adjoint, non-negative operator by the spectral theorem.

Note that we have

$$\operatorname{dom} A \subset d[a]\,, \tag{2.20}$$
$$a[u,v] = (Au,v) \qquad \text{for all } u \in \operatorname{dom} A\,,\ v \in d[a]\,. \tag{2.21}$$

Moreover, the quadratic form a is lower semibounded with $m_a = m_A$. Indeed, by dropping a non-negative term in the definition of a we find $a[u] \geq m_A \|u\|^2$ for all $u \in d[a]$, which implies $m_a \geq m_A$, and conversely, since $a[u] = (Au,u)$ for $u \in \operatorname{dom} A$ by (2.21), we see by enlarging the set over which the infimum is taken that $m_A \geq m_a$.

The quadratic form a is closed. This follows from the fact that $(A+m)^{1/2}$ is closed.

We now return to general quadratic forms. A lower semibounded, closed quadratic form a satisfying (2.20) and (2.21) is called *associated to the operator* A. Since $\operatorname{dom} A$ is dense in $d[a]$ with respect to the norm $(a[u] + m\|u\|^2)^{1/2}$, the associated quadratic form of a lower semibounded, self-adjoint operator is unique and coincides with the one constructed above. In particular, if a is associated to A, then $m_a = m_A$.

The usefulness of lower semi-bounded quadratic forms comes from the fact that the above construction can be reversed, that is, each closed, lower semibounded quadratic form defines a unique self-adjoint operator. More precisely, we have

Theorem 2.15. *Let a be a lower semibounded, closed quadratic form. Then there is a unique, lower semibounded and self-adjoint operator A which is associated to a. Moreover,* $m_A = m_a$ *and the domain of A is given by*

$$\operatorname{dom} A = \{u \in d[a] :\ \exists f \in \mathscr{H} \text{ such that } \forall\, v \in d[a]\,,\ a[u,v] = (f,v)\}\,.$$

The proof is based on Riesz' representation theorem and is omitted.

We point out that the quadratic form a can be represented via the spectral measure P_λ of the operator A as follows,

$$a[u,v] = \int_{\mathbb{R}} \lambda\, d(P_\lambda u, v) \qquad \text{for all } u,v \in d[a]\,. \tag{2.22}$$

This is a consequence of (2.8), the identity (2.19) and the standard polarization argument.

It is convenient to describe spectral properties of A in terms of the quadratic form a.

Lemma 2.16. *Let a be a lower semi-bounded, closed quadratic form and A the corresponding operator as in Theorem 2.15. Then*

$$m_a = \inf \sigma(A)\,.$$

Moreover, m_a *is an eigenvalue of A if and only if the infimum* $\inf_{0 \neq u \in d[a]} a[u]/\|u\|^2$ *is a minimum, and eigenvectors are precisely those vectors for which the infimum is attained.*

Proof. The equality follows immediately from $m_a = m_A$ and Lemma 2.8. Moreover, given the characterization of minimizers there, it remains to be shown that if $u \in d[a]$

is a minimizer for $\inf_{0\neq u\in d[a]} a[u]/\|u\|^2$, then $u \in \operatorname{dom} A$. This follows from (2.22), which implies $u = P_{\{m_a\}}u$ and therefore $u \in \operatorname{dom} A$. □

We say that A has discrete spectrum if and only if the essential spectrum of A is empty. This is equivalent to the fact that for any compact interval $\delta \subset \mathbb{R}$ we have $\dim P_\delta \mathscr{H} < \infty$.

The following lemma gives a necessary and sufficient condition for a lower semibounded operator to have discrete spectrum in terms of the *embedding operator* $\mathscr{J} : d[a] \to \mathscr{H}$ which maps every $u \in d[a]$ to itself as an element in $\mathscr{H}$. This is a bounded operator when $d[a]$ is equipped with its norm $\sqrt{a[u]+m\|u\|^2}$ for some $m > -m_a$. By definition, this operator is compact if the closed unit ball in $d[a]$ is a relatively compact set in $\mathscr{H}$.

Lemma 2.17. *Let A be a self-adjoint and lower semibounded operator. Then A has discrete spectrum if and only if the embedding from $d[a]$ to $\mathscr{H}$ is compact.*

Proof. Fix $m > -m_a$ and define the norm on $d[a]$ with respect to this m. The embedding operator $d[a] \ni u \mapsto u \in \mathscr{H}$ is compact if and only if the unit ball $\sqrt{a[u]+m\|u\|^2} \leq 1$ is a relatively compact subset of $\mathscr{H}$. Setting $u = (A+m)^{-1/2}v$, we see that this is equivalent to the compactness of $(A+m)^{-1/2}$ in $\mathscr{H}$. Finally, by Lemmas 2.11 and 2.12, $(A+m)^{-1/2}$ is compact if and only if $\{0\} = \sigma_{\mathrm{ess}}((A+m)^{-1/2})$. By the spectral mapping theorem, see (2.13), this property holds, if and only if the spectrum of A is discrete. This completes the proof. □

Remark 2.18. As in the proof of Lemma 2.1, one can show that the embedding from $d[a]$ to $\mathscr{H}$ is compact if and only if every weakly convergent sequence in $d[a]$ converges strongly in $\mathscr{H}$.

2.2.2 *The operators T^*T and TT^*. Polar decomposition of an operator*

Let T be a densely defined, closed operator in a Hilbert space $\mathscr{H}$. Consider the quadratic form $a[u] = \|Tu\|^2$ defined on $d[a] = \operatorname{dom} T$. This form is closed and non-negative. Hence, by Theorem 2.15 it induces a self-adjoint non-negative operator A with

$$\operatorname{dom} A = \{u \in \operatorname{dom} T : \exists f \in \mathscr{H} \text{ such that } \forall\, v \in \operatorname{dom} T\,,\ (Tu, Tv) = (f, v)\}\ .$$

This means $Tu \in \operatorname{dom} T^*$ and $Au = T^*Tu$ for all $u \in \operatorname{dom} A$, which means that $A = T^*T$ in the sense of composition of unbounded operators.

Since T^*T is self-adjoint and non-negative, its square root is defined by the spectral theorem. The operator

$$|T| = (T^*T)^{1/2}$$

is called the *absolute value* of T. It is the unique self-adjoint, non-negative operator on $\mathscr{H}$ with

$$\||T|f\| = \|Tf\| \quad \text{for any} \quad f \in \operatorname{dom} T = \operatorname{dom}|T|\,. \tag{2.23}$$

Indeed, since by its definition $|T|^2 = T^*T$, the associated quadratic forms $\||T|f\|^2$ and $\|Tf\|^2$ coincide, including equality of their respective domains. The uniqueness follows from the uniqueness of the positive square root of a non-negative operator and the uniqueness in Theorem 2.15.

Any complex number z has a polar representation $z = e^{i\varphi}r$ with $r = |z| \geq 0$ and $\varphi = \arg z \in [0, 2\pi)$. In this subsection we present an analogous representation of an operator in a Hilbert space.

The operator $|T|$ will correspond to the 'radial part' in the polar decomposition of T. The following theorem describes the 'angular part' of this decomposition.

Proposition 2.19. *Let T be a densely defined, closed operator. Then there is a unique bounded operator U such that $T = U|T|$ and*

$$\|Uf\| = \|f\| \quad \textit{for all} \quad f \in (\ker T)^\perp\,, \tag{2.24}$$

$$\ker U = \ker T\,, \tag{2.25}$$

$$\operatorname{ran} U = \overline{\operatorname{ran} T}\,. \tag{2.26}$$

Proof. We define $U : \operatorname{ran}|T| \to \operatorname{ran} T$ by $U|T|f = Tf$. According to (2.23) the operator U is well-defined, norm preserving and maps onto $\operatorname{ran} T$. Thus, U extends to a unitary operator from $\overline{\operatorname{ran}|T|}$ to $\overline{\operatorname{ran} T}$ and, extending U by zero to $(\operatorname{ran}|T|)^\perp$, we obtain a bounded operator on $\mathscr{H}$ satisfying (2.26).

To prove (2.24) and (2.25) it remains to show that $\overline{\operatorname{ran}|T|} = (\ker T)^\perp$. Indeed, using the self-adjointness of $|T|$, we have $\overline{\operatorname{ran}|T|} = (\ker|T|)^\perp$. Again by (2.23) we have $\ker|T| = \ker T$. This proves the existence of U with the claimed properties.

Uniqueness of U comes from the fact that the extension of U from $\operatorname{ran}|T|$ to $\overline{\operatorname{ran}|T|}$ is unique. □

Because T is closed, we have $T^{**} = T$, and the same construction as at the beginning of this subsection leads to the self-adjoint operator TT^* defined on $\{u \in \operatorname{dom} T^* : T^*u \in \operatorname{dom} T\}$ being associated to the quadratic form $\|T^*u\|^2$ defined on $\operatorname{dom} T^*$. Our next result compares the operators T^*T and TT^* away from their respective kernels.

Proposition 2.20. *Let T be a densely defined, closed operator. The operator T^*T restricted to $\ker(T^*T)^\perp$ is unitarily equivalent to the operator TT^* restricted to $\ker(TT^*)^\perp$.*

Proof. Let $T = U|T|$ be the polar decomposition of T. First we observe, that $T^* = |T|U^*$ with $\operatorname{dom} T^* = \{f \in \mathscr{H} : \ U^*f \in \operatorname{dom}|T|\}$. Indeed, by the definition of the adjoint we have

$$\begin{aligned}\operatorname{dom} T^* = &\{f \in \mathscr{H} : \text{ there is a } g \in \mathscr{H} \text{ such that} \\ &\qquad \text{for all } h \in \operatorname{dom}|T| \text{ one has } (g,h) = (U^*f, |T|h)\} \\ = &\{f \in \mathscr{H} : \ U^*f \in \operatorname{dom}|T|^*\}\,.\end{aligned}$$

Since $|T|^* = |T|$ this means $\operatorname{dom}|T|U^* = \operatorname{dom}(T^*)$ and $T^*f = |T|U^*f$.

Note that because of $\operatorname{dom}|T| = \operatorname{dom}T$, we also have $\operatorname{dom}T^* = \operatorname{dom}TU^*$. By the formula for T^* and by (2.23) we have

$$\|T^*u\| = \||T|U^*u\| = \|TU^*u\| \quad \text{for all} \quad u \in \operatorname{dom}T^* = \operatorname{dom}TU^*\,.$$

For the associated self-adjoint operators this means $TT^* = (TU^*)^*TU^*$. The same argument as before implies $(UT^*)^* = TU^*$. Since UT^* is closed, this gives $(TU^*)^* = UT^*$ and therefore

$$TT^* = UT^*TU^*\,. \tag{2.27}$$

According to Proposition 2.19 the operator U restricts to a unitary operator $V : (\ker T)^\perp \to \overline{\operatorname{ran}T}$. Consequently, we have an unitary operator $V^* : \overline{\operatorname{ran}T} \to (\ker T)^\perp$. Since $\ker TT^* = \ker T^* = (\operatorname{ran}T)^\perp$ and $\ker T^*T = \ker T$, we can restrict (2.27) to

$$TT^*|_{(\ker TT^*)^\perp} = V\left(T^*T|_{(\ker T^*T)^\perp}\right)V^*\,. \tag{2.28}$$

This provides the claimed unitary equivalence. □

Remark 2.21. A particular consequence of Proposition 2.20 is that for any real $\lambda \neq 0$ the dimension of the eigenspaces of T^*T and TT^* corresponding to λ coincide.

Problem 2.22. Show that if u is an eigenvector for T^*T corresponding to an eigenvalue $\lambda \neq 0$, then Tu is non-zero and an eigenvector for TT^* corresponding to eigenvalue λ.

2.2.3 The variational principle

We begin with a version of the variational principle that is sometimes called *Glazman's lemma*. Let A be self-adjoint. We introduce the quantity

$$N(\mu, A) = \dim P_{(-\infty,\mu)}\mathscr{H}\,, \quad \mu \in \mathbb{R}\,,$$

which can be a natural number or infinite. If for a given $\mu \in \mathbb{R}$ it is finite, then the spectrum of A below μ consists of finitely many eigenvalues with finite multiplicities only, and $N(\mu, A)$ equals the total multiplicity of these eigenvalues. It is referred to as the *spectral counting function*.

Theorem 2.23. *Let A be a self-adjoint, lower semibounded operator. Then for any $\mu \in \mathbb{R}$*

$$\begin{aligned} N(\mu, A) = \sup\{\dim F :\ & F \subset d[a] \textit{ is a subspace such that} \\ & \textit{for all } 0 \neq u \in F \textit{ one has } a[u] < \mu\|u\|^2\} \end{aligned} \tag{2.29}$$

and

$$N(\mu,A)+\dim\ker(A-\mu)=\sup\{\dim F:\ F\subset d[a] \textit{ is a subspace such that } \textit{for all } u\in F \textit{ one has } a[u]\leq\mu\|u\|^2\}. \qquad (2.30)$$

If $\mathscr{F}$ is a form core of a, then in (2.29) *it suffices to take the supremum over $F\subset\mathscr{F}$ only.*

In particular, if the right side in (2.29) is finite, then it coincides with the number of eigenvalues of A which are strictly less than μ (counting multiplicities).

Proof. Let $F=P_{(-\infty,\mu)}\mathscr{H}$ and note that this is contained in $d[a]$. Moreover, for any $0\neq u\in F$, by the spectral theorem,

$$a[u]=\int_{(-\infty,\mu)}\lambda\,d(P_\lambda u,u)<\mu\int_{(-\infty,\mu)}d(P_\lambda u,u)=\mu\|u\|^2,$$

where P denotes the spectral measure of A. This proves "$\leq$" in (2.29).

To prove the reverse inequality, we consider an arbitrary subspace $F\subset d[a]$ with $\dim F>N(\mu,A)$. Since $\dim P_{(-\infty,\mu)}\mathscr{H}=N(\mu,A)$, there is a $0\neq u_0\in F\cap\big(P_{(-\infty,\mu)}\mathscr{H}\big)^\perp$. Then, again by the spectral theorem, we have

$$a[u_0]=\int_{[\mu,\infty)}\lambda\,d(P_\lambda u_0,u_0)\geq\mu\int_{[\mu,\infty)}d(P_\lambda u_0,u_0)=\mu\|u_0\|^2.$$

Therefore this subspace F is not admissible in the right side of (2.29).

An analogous argument proves the second identity (2.30).

The proof when F is restricted to lie in a dense subspace $\mathscr{F}$ of $d[a]$ follows from approximating in the form norm a finite linear system from $P_{(-\infty,\mu)}\mathscr{H}$ by elements from $\mathscr{F}$. This preserves the strict inequality $a[u]<\mu\|u\|^2$, $u\neq 0$, on the span of these elements and completes the proof. □

Sometimes the following modification of Glazman's lemma is useful.

Theorem 2.24. *Let A be a self-adjoint, lower semibounded operator. Then for any $\mu\in\mathbb{R}$*

$$N(\mu,A)=\inf\{\dim F:\ F\subset\mathscr{H} \textit{ is a subspace such that } \textit{for all } u\in F^\perp\cap d[a],\ a[u]\geq\mu\|u\|^2\} \qquad (2.31)$$

and

$$N(\mu,A)+\dim\ker(A-\mu)=\inf\{\dim F:\ F\subset\mathscr{H} \textit{ is a subspace such that } \textit{for all } 0\neq u\in F^\perp\cap d[a],\ a[u]>\mu\|u\|^2\}. \qquad (2.32)$$

Proof. For $F=P_{(-\infty,\mu)}\mathscr{H}$ we have $N(\mu,A)=\dim F$ and by the spectral theorem $a[u]\geq\mu\|u\|^2$ for all $u\in F^\perp\cap d[a]$. This proves "$\geq$" in (2.31).

To prove equality, we consider some arbitrary subspace $F \subset \mathscr{H}$ with $N(\mu, A) > \dim F$. Then there exists some $u_0 \neq 0$ with $u_0 \in P_{(-\infty,\mu)}\mathscr{H} \subset d[a]$ and $u_0 \perp F$. But for this u_0 the opposite inequality $a[u_0] < \mu \|u_0\|^2$ holds true. Hence, this subspace is not admissible in the right side of (2.31).

An analogous argument proves the second identity (2.32). □

The variational characterization of the counting function of an operator is equivalent to a variational characterization of its individual eigenvalues below the bottom of its essential spectrum. For the explicit formulation and a discussion of this principle we refer to Section 2.3.

Next, we discuss some important applications of Glazman's lemma. We begin by introducing the important notion of comparison of two operators. Let A and B be self-adjoint, lower semibounded with associated quadratic forms a and b with the form domains $d[a]$ and $d[b]$, respectively.

Definition 2.25. We say that A is *greater or equal than* B, in symbols $A \geq B$, if the following two conditions are satisfied

$$d[a] \subset d[b]\,, \tag{2.33}$$

$$a[x,x] \geq b[x,x] \quad \text{for all} \quad x \in d[a]\,. \tag{2.34}$$

Note that the case $d[a] = d[b]$ in (2.33) can occur, for instance, if both operators A and B are bounded. On the other hand the definition is meaningful, if we have the special case $a[x,x] = b[x,x]$ for all $x \in d[a]$ in (2.34). In applications this occurs, for instance, for differential operators, when the non-trivial embedding (2.33) stands for different choices of boundary conditions for A and B.

Trivially, it holds $A \geq A$. Further note that $A \geq B$ and $B \geq C$ implies $A \geq C$. Finally, if $A \geq B$ and $B \geq A$ then the forms a and b coincide on $d[a] = d[b]$ and $A = B$. Therefore this comparison defines an order relation.

Below we shall also use the notation $A \leq B$ in the obvious sense of $B \geq A$.

The following is an immediate consequence of Glazman's lemma (Theorem 2.23).

Proposition 2.26. *Let A and B be self-adjoint, lower semibounded operators satisfying $A \geq B$. Then*

$$N(\mu, A) \leq N(\mu, B) \quad \textit{for all} \quad \mu \in \mathbb{R}\,.$$

In particular, if the spectrum of B is discrete in $(-\infty, \kappa)$ for some $\kappa \leq \infty$, then the spectrum of A in $(-\infty, \kappa)$ is discrete as well and $\lambda_n(A) \geq \lambda_n(B)$ for all n with $\lambda_n(A) < \kappa$.

The second application of Glazman's lemma concerns eigenvalues of sums of operators defined in the form sense. Consider two self-adjoint, lower semi-bounded operators A and B with associated closed quadratic forms a and b. Assume that $d[a] \cap d[b]$ is dense in $\mathscr{H}$. Then let c be the quadratic form $c[u] = a[u] + b[u]$ with domain $d[c] = d[a] \cap d[b]$. Again, this form is semibounded from below and closed.

It induces an associated self-adjoint operator C, which we formally denote by $A+B$. The identification of C with $A+B$ has to be understood in the form sense, not in the sense of a sum of operators.

Corollary 2.27. *Let A and B be self-adjoint, lower semibounded operators with associated quadratic forms a and b, and assume that $d[a]\cap d[b]$ is dense in $\mathscr{H}$. Then*

$$N(0,A+B) \leq N(0,A)+N(0,B)\,.$$

Proof. Let $P(A)$ and $P(B)$ denote the spectral measures for A and B, respectively, and let $L = P_{(-\infty,0)}(A)\mathscr{H}$ and $M = P_{(-\infty,0)}(B)\mathscr{H}$. By the spectral theorem,

$$a[u] \geq 0 \qquad \text{provided } u \in d[a]\cap L^{\perp}$$

and

$$b[v] \geq 0 \qquad \text{provided } v \in d[b]\cap M^{\perp}\,.$$

Thus,

$$c[w] = a[w]+b[w] \geq 0 \qquad \text{provided } w \in d[a]\cap d[b]\cap (L+M)^{\perp}\,.$$

By Theorem 2.24 we have

$$N(0,C) \leq \dim(L+M) \leq \dim L + \dim M = N(0,A)+N(0,B)\,,$$

as claimed. □

The following application of Glazman's lemma is of a more technical nature and will be needed in the following subsection. Let A be a self-adjoint, lower semibounded operator and let a be the associated quadratic form. Consider a bounded operator T such that

$$d[a_T] = \{u \in \mathscr{H} : Tu \in d[a]\}$$

is dense in $\mathscr{H}$ and define $a_T[u] = a[Tu]$ for $u \in d[a_T]$. This form is bounded from below and closed. It induces the associated self-adjoint operator A_T, which we also denote by T^*AT. The identification of A_T with T^*AT has to be understood in the form sense, not in the sense of a product of operators.

Corollary 2.28. *If A is self-adjoint and lower semi-bounded and if T is bounded, then $N(0,T^*AT) \leq N(0,A)$.*

Proof. Put $L = P_{(-\infty,0)}\mathscr{H}$, where P is the spectral measure of A. Then for all $u \in (T^*L)^{\perp}\cap d[a_T]$ one has $Tu \in L^{\perp}\cap d[a]$. Thus, by the spectral theorem we have $a_T[u] = a[Tu] \geq 0$. By Glazman's lemma, see Theorem 2.24, this implies

$$N(0,T^*AT) \leq \dim T^*L \leq \dim L = N(0,A)\,,$$

as claimed. □

We shall also need the following version of the previous corollary. Let P be an orthogonal projection on $\mathscr{H}$ and $\mathscr{H}_P = P\mathscr{H}$. As above, let A be a self-adjoint lower semibounded operator on $\mathscr{H}$ with associated quadratic form a. Provided that $\operatorname{ran} P \subset d[a]$ let $A_P = PAP$ to be understood in the form sense as explained above as an operator on $\mathscr{H}$. Let $\tilde{A}_P$ be the restriction of A_P to $\mathscr{H}_P$. We now compare the spectrum of A on $\mathscr{H}$ with that of $\tilde{A}_P$ on $\mathscr{H}_P$.

Corollary 2.29. *If A is self-adjoint and lower semibounded and if P is a projection, then $N(\lambda,\tilde{A}_P) \leq N(\lambda,A)$ for all $\lambda \in \mathbb{R}$.*

Proof. Let $B = A - \lambda$ in $\mathscr{H}$. Then by Corollary 2.28 $N(\lambda,A) = N(0,B) \geq N(0,B_P)$. Since $N(0,B_P)$ counts the strictly negative eigenvalues and $B_P = \tilde{B}_P \oplus \mathbb{O}$ on $\mathscr{H} = \mathscr{H}_P \oplus \mathscr{H}_P^{\perp}$, we get $N(0,B_P) = N(0,\tilde{B}_P)$. On $\mathscr{H}_P$ we have $\tilde{B}_P = \tilde{A}_P - \lambda$ and therefore $N(0,\tilde{B}_P) = N(\lambda,\tilde{A}_P)$. □

2.2.4 Variational principle for sums of eigenvalues

Let A be a self-adjoint lower semibound operator. Then the spectrum of A below $\kappa := \inf \sigma_{\text{ess}}(A) \in (-\infty,\infty]$ is discrete. If it is not empty, this portion of the spectrum consists of eigenvalues of finite multiplicities which may accumulate at the value κ only. Therefore these eigenvalues can be enumerated in increasing order

$$\lambda_1(A) \leq \lambda_2(A) \leq \ldots \leq \lambda_k(A) \leq \ldots,$$

where each eigenvalue is repeated according to its multiplicity. This list contains $N(\kappa,A)$ elements and may be empty, finite or infinite.

Here is a variational characterization for partial sums of these eigenvalues.

Proposition 2.30. *Let A be a self-adjoint, lower semibounded operator with associated quadratic form a and let $\kappa := \inf \sigma_{\text{ess}}(A) \in (-\infty,\infty]$. Then for any finite number $N \leq N(\kappa,A)$ it holds*

$$\sum_{j=1}^{N} \lambda_j(A) = \inf \left\{ \sum_{j=1}^{N} a[u_j] : \ (u_j)_{j=1}^{N} \subset d[a] \text{ and } (u_j,u_k) = \delta_{j,k} \quad \forall 1 \leq j,k \leq N \right\}.$$

Proof. The inequality ”$\geq$” follows immediately by choosing u_j to be orthonormal eigenvectors corresponding to $\lambda_j(A)$. Let us prove the opposite bound. For a given finite number $N \leq N(\kappa,A)$ and given $u_1,\ldots,u_N \in d[a]$ with $(u_j,u_k) = \delta_{j,k}$ we define the orthogonal projection $P = \sum_{j=1}^{N} (\cdot,u_j)u_j$. Then, according to Corollary 2.29,

$$\lambda_j(A) \leq \lambda_j(\tilde{A}_P) \qquad \text{for all } 1 \leq j \leq N.$$

Thus,

$$\sum_{j=1}^{N} \lambda_j(A) \leq \sum_{j=1}^{N} \lambda_j\left(\tilde{A}_P\right) = \sum_{k=1}^{N} a[u_k],$$

where we use the fact from linear algebra that the sum of all eigenvalues of a matrix in a finite dimensional space can be evaluated by the sum of the diagonal entries. This proves the proposition. □

Remark 2.31. `orthonormality can be relaxed to sub-orthonormality`

Proposition 2.30 gives rise to the following somewhat technical result, which will be of importance in Section 7.4. Let $(\Xi, d\xi)$ be a probability space and let U be a measurable function on Ξ taking values $U(\xi)$ in the unitary operators on $\mathscr{H}$. Measurability is understood in the form sense. Let A be a self-adjoint, lower semibounded operator with an associated quadratic form a. We assume that $\operatorname{ran} U(\xi) \subset d[a]$ for a.e. $\xi \in \Xi$. Assume that the quadratic form

$$a^U[u] = \int_\Xi a[U(\xi)u]\, d\xi$$

is densely defined on $d[a^U] = \{u \in \mathscr{H} : a^U[u] < \infty\}$. This form is semibounded from below and closed. It defines an associated self-adjoint operator, which we denote by $A^U = \int_\Xi U^*(\xi) A U(\xi)\, d\xi$.

Corollary 2.32. *Let A be a self-adjoint, lower semibounded operator with the associated quadratic form a and let $\kappa_A := \inf \sigma_{\mathrm{ess}}(A)$ and $\kappa_{A^U} := \inf \sigma_{\mathrm{ess}}(A^U)$. Then for any finite number $N \le \min\{N(\kappa_A, A), N(\kappa_{A^U}, A^U)\}$ it holds*

$$\sum_{n=1}^N \lambda_n(A^U) \ge \sum_{n=1}^N \lambda_n(A)\,.$$

Proof. Let $(u_j)_{j=1}^N$ be a orthonormal system of eigenvectors of A^U corresponding to the eigenvalues $\lambda_1(A^U), \cdots, \lambda_N(A^U)$. Then

$$\sum_{n=1}^N \lambda_n(A^U) = \sum_{j=1}^N a^U[u_j] = \sum_{j=1}^N \int_\Xi a[U(\xi)u_j]\, d\xi = \int_\Xi \sum_{j=1}^N a[U(\xi)u_j]\, d\xi\,.$$

Since for a.e. $\xi \in \Xi$ the system $(U(\xi)u_j)_{j=1}^N$ belongs to $d[a]$ and is orthonormal, the variational principle from Proposition 2.30 implies

$$\sum_{j=1}^N a[U(\xi)u_j] \ge \sum_{n=1}^N \lambda_n(A)\,.$$

Since $d\xi$ is a probability measure, it remains to integrate both sides in this inequality. □

The next result is a variant of Proposition 2.30 where N is not fixed. For this purpose the following notion of a trace of a non-negative bounded operator T will be useful. If the essential spectrum of T is empty or consists of the point 0 only, the spectrum of $-T$ below 0 is discrete and consists of the eigenvalues $\lambda_n(-T)$, which are enumerated counting multiplicities as explained above. We put

$$\operatorname{Tr} T = \begin{cases} +\infty & \text{if} \quad \sup \sigma_{\text{ess}}(T) > 0\,, \\ -\sum_n \lambda_n(-T) & \text{otherwise}\,. \end{cases} \tag{2.35}$$

Even in the second case this value can be infinite. On the other hand, if $\operatorname{Tr} T$ is finite, then Lemma 2.12 implies that T is compact.

Corollary 2.33. *Let A be a self-adjoint, lower semibounded operator. Then*

$$-\operatorname{Tr} A_- = \inf \left\{ \sum_{j=1}^{N} a[u_j] :\ N \in \mathbb{N},\ (u_j)_{j=1}^{N} \subset d[a] \text{ and } (u_j, u_k) = \delta_{j,k} \forall 1 \le j,k \le N \right\}.$$

Proof. If $\inf \sigma_{\text{ess}}(A) \in [0,\infty]$, then

$$-\operatorname{Tr} A_- = \inf_{N \in \mathbb{N}} \sum_{j=1}^{N} \lambda_j(A)\,,$$

and therefore the corollary follows from Proposition 2.30. On the other hand, if $\kappa = \inf \sigma_{\text{ess}}(A) \in (-\infty, 0)$, then $\operatorname{Tr} A_- = \infty$ and $\dim P_{(-\infty,\kappa+\varepsilon)} \mathscr{H} = \infty$ for any $\varepsilon > 0$. In particular, if $\kappa + \varepsilon < 0$, there is an infinite sequence of orthonormal (u_j) with $a[u_j] \le \kappa + \varepsilon$. Thus, the right hand side in the corollary is also equal to $-\infty$. □

Lemma 2.34. *Assume that T is bounded and non-negative. Then for any complete orthonormal system (u_j),*

$$\operatorname{Tr} T = \sum_j (T u_j, u_j)\,.$$

We emphasize that the lemma does not assume that $\operatorname{Tr} T$ is finite or even that T is compact.

Proof. We first assume that $\operatorname{Tr} T < \infty$ and we choose a complete orthonormal system (v_n) such that $T v_n = -\lambda_n(-T) v_n$. Then

$$\begin{aligned} \operatorname{Tr} T &= -\sum_n \lambda_n(-T) = \sum_n \|T^{1/2} v_n\|^2 = \sum_n \sum_j |(T^{1/2} v_n, u_j)|^2 = \sum_j \sum_n |(v_n, T^{1/2} u_j)|^2 \\ &= \sum_j \|T^{1/2} u_j\|^2 = \sum_j (T u_j, u_j)\,, \end{aligned}$$

as claimed. (The interchange of summations here is justified since all terms are non-negative.) On the other hand, when $\operatorname{Tr} T = \infty$, by Corollary 2.33, for any $M > 0$ there is an $N \in \mathbb{N}$ and an orthonormal system $(v_n)_{n=1}^{N}$ such that $\sum_{n=1}^{N} \|T^{1/2} v_n\|^2 \ge M$. Similarly as before, we have

$$\begin{aligned} \sum_{n=1}^{N} \|T^{1/2} v_n\|^2 &= \sum_{n=1}^{N} \sum_j |(T^{1/2} v_n, u_j)|^2 = \sum_j \sum_{n=1}^{N} |(v_n, T^{1/2} u_j)|^2 \\ &\le \sum_j \|T^{1/2} u_j\|^2 = \sum_j (T u_j, u_j)\,. \end{aligned}$$

Since M is arbitrary, this means that $\sum_j (Tu_j, u_j) = \infty$.

The next corollary is the analogue of Corollary 2.27 for sums of eigenvalues.

Corollary 2.35. *Let A and B be self-adjoint, lower semibounded operators with associated quadratic forms a and b, and assume that $d[a] \cap d[b]$ is dense in $\mathscr{H}$ and let $A+B$ be defined in the form sense. If* $\operatorname{Tr} A_- + \operatorname{Tr} B_- < \infty$*, then* $\operatorname{Tr}(A+B)_- < \infty$ *and*

$$\operatorname{Tr}(A+B)_- \leq \operatorname{Tr} A_- + \operatorname{Tr} B_- .$$

Proof. Let $N \in \mathbb{N}$ and let $(u_j)_{j=1}^N \subset d[a] \cap d[b]$ be orthonormal functions. Then, by Corollary 2.33 for the operators A and B,

$$\sum_{j=1}^N (a+b)[u_j] = \sum_{j=1}^N a[u_j] + \sum_{j=1}^N b[u_j] \geq -\operatorname{Tr} A_- - \operatorname{Tr} B_-$$

Taking the infimum over all N and $(u_j)_{j=1}^N$ as above, we obtain again from Corollary 2.33 the inequality of the statement. ☐

2.2.5 Riesz means

In this section we are interested in so-called *Riesz means* of order $\gamma > 0$ of a self-adjoint, lower semibounded operator A, namely, the quantities

$$\operatorname{Tr} A_-^\gamma .$$

If this quantity is finite, then the negative spectrum of A consists of eigenvalues of finite multiplicities and, if $\lambda_1(A) \leq \lambda_2(A) \leq \ldots$ is an enumeration of those, we have

$$\operatorname{Tr} A_-^\gamma = \sum_n \lambda_n(A)_-^\gamma .$$

A very useful identity connects the Riesz means to the spectral counting function $N(-\kappa, A)$ for $\kappa > 0$.

Lemma 2.36. *Let A be a self-adjoint, lower semibounded operator and let $\gamma > 0$. Then*

$$\operatorname{Tr} A_-^\gamma = \gamma \int_0^\infty N(-\kappa, A)\, \kappa^{\gamma-1}\, d\kappa .$$

Proof. Since

$$a_-^\gamma = \gamma \int_0^\infty \chi_{\{a < -\kappa\}} \kappa^{\gamma-1}\, d\kappa \qquad \text{for } a \in \mathbb{R},$$

the identity follows from the spectral theorem. ☐

One of the consequences of this formula is a generalization of Corollary 2.27 to the case of Riesz means.

Proposition 2.37. *Let A and B be self-adjoint, lower semibounded operators with associated quadratic forms a and b, and assume that $d[a] \cap d[b]$ is dense in $\mathscr{H}$ and let $A+B$ be defined in the form sense. If $\gamma > 0$ and if $\operatorname{Tr} A_-^\gamma + \operatorname{Tr} B_-^\gamma < \infty$, then $\operatorname{Tr}(A+B)_-^\gamma < \infty$ and for any $0 < \theta < 1$*

$$\operatorname{Tr}(A+B)_-^\gamma \leq \theta^{-\gamma} \operatorname{Tr} A_-^\gamma + (1-\theta)^{-\gamma} \operatorname{Tr} B_-^\gamma .$$

Proof. According to Corollary 2.27, for any $\tau > 0$ and for any $0 < \theta < 1$ we have the bound

$$\begin{aligned} N(-\kappa, A+B) &= N(0, (A+\theta\kappa) + (B + (1-\theta)\kappa)) \\ &\leq N(0, A+\theta\kappa) + N(0, B+(1-\theta)\kappa) \\ &= N(-\theta\kappa, A) + N(-(1-\theta)\kappa, B) . \end{aligned}$$

Therefore, by Lemma 2.36

$$\begin{aligned} \operatorname{Tr}(A+B)_-^\gamma &= \gamma \int_0^\infty N(-\kappa, A+B) \kappa^{\gamma-1} \, d\kappa \\ &\leq \gamma \int_0^\infty N(-\theta\kappa, A) \kappa^{\gamma-1} \, d\kappa \\ &\qquad + \gamma \int_0^\infty N(-(1-\theta)\kappa, B) \kappa^{\gamma-1} \, d\kappa \\ &= \theta^{-\gamma} \operatorname{Tr} A_-^\gamma + (1-\theta)^{-\gamma} \operatorname{Tr} B_-^\gamma , \end{aligned}$$

as claimed. □

The remainder of this subsection contains improvements of the constants appearing in the bound in Proposition 2.37. These improved constants are not necessary for the applications in this book and can be omitted in a first reading.

The fact that an improvement is possible can be seen from Corollary 2.35, which contains a bound like that in Proposition 2.37, but without the prefactors $\theta^{-\gamma}$ and $(1-\theta)^{-\gamma}$.

The first improvements concerns the case $\gamma > 1$. In this case we start from the following variation of Lemma 2.36.

Lemma 2.38. *Let A be a self-adjoint, lower semibounded operator and let $\gamma > 1$. Then*

$$\operatorname{Tr} A_-^\gamma = \gamma(\gamma-1) \int_0^\infty \operatorname{Tr}(A+\kappa)_+ \, \kappa^{\gamma-2} \, d\kappa .$$

Proof. Since

$$a_-^\gamma = \gamma(\gamma-1) \int_0^\infty (a+\kappa)_- \kappa^{\gamma-1} \, d\kappa \qquad \text{for } a \in \mathbb{R} ,$$

the identity follows from the spectral theorem. □

Using this lemma we obtain the following bound.

Proposition 2.39. *Let A and B be self-adjoint, lower semibounded operators with associated quadratic forms a and b, and assume that $d[a]\cap d[b]$ is dense in $\mathscr{H}$ and let $A+B$ be defined in the form sense. If $\gamma>1$ and if $\operatorname{Tr}A_-^\gamma+\operatorname{Tr}B_-^\gamma<\infty$, then $\operatorname{Tr}(A+B)_-^\gamma<\infty$ and for any $0<\theta<1$*

$$\operatorname{Tr}(A+B)_-^\gamma \leq \theta^{-\gamma+1}\operatorname{Tr}A_-^\gamma+(1-\theta)^{-\gamma+1}\operatorname{Tr}B_-^\gamma\,.$$

Proof. According to what we have shown in the first part of the proof, for any $\tau>0$ and for any $0<\theta<1$ we have the bound

$$\begin{aligned}\operatorname{Tr}(A+B+\kappa)_- &= \operatorname{Tr}\left((A+\theta\kappa)+(B+(1-\theta)\kappa)\right)_- \\ &\leq \operatorname{Tr}(A+\theta\kappa)_-+\operatorname{Tr}(B+(1-\theta)\kappa)_-\,.\end{aligned}$$

Therefore

$$\begin{aligned}\operatorname{Tr}(A+B)_-^\gamma &= \gamma(\gamma-1)\int_0^\infty \operatorname{Tr}(A+B+\kappa)_-\,\kappa^{\gamma-2}\,d\kappa \\ &\leq \gamma(\gamma-1)\int_0^\infty \operatorname{Tr}(A+\theta\kappa)_-\,\kappa^{\gamma-2}\,d\kappa \\ &\quad+\gamma(\gamma-1)\int_0^\infty \operatorname{Tr}(B+(1-\theta)\kappa)_-\,\kappa^{\gamma-2}\,d\kappa \\ &= \theta^{-\gamma+1}\operatorname{Tr}A_-^\gamma+(1-\theta)^{-\gamma+1}\operatorname{Tr}B_-^\gamma\,,\end{aligned}$$

as claimed. □

Even for numbers $a,b\in\mathbb{R}$ constant in the inequality

$$(a+b)_-^\gamma \leq \theta^{-\gamma+1}a_-^\gamma+(1-\theta)^{-\gamma+1}b_-^\gamma$$

cannot be improved for $\gamma>1$. On the other hand, for $0<\gamma<1$, the inequality for numbers holds without the factors of $\theta^{-\gamma+1}$ and $(1-\theta)^{-\gamma+1}$. This motivates the next result, which is an improvement of Proposition 2.37 in the case $0<\gamma<1$. It is a special case of a theorem of Rotfel'd [Rotfel'd(1967)], [Rotfel'd(1968)]. `correct? which one or both? Reference! 1967`

Proposition 2.40. *Let A and B be self-adjoint, lower semibounded operators with associated quadratic forms a and b, and assume that $d[a]\cap d[b]$ is dense in $\mathscr{H}$ and let $A+B$ be defined in the form sense. If $0<\gamma<1$ and if $\operatorname{Tr}A_-^\gamma+\operatorname{Tr}B_-^\gamma<\infty$, then $\operatorname{Tr}(A+B)_-^\gamma<\infty$ and*

$$\operatorname{Tr}(A+B)_-^\gamma \leq \operatorname{Tr}A_-^\gamma+\operatorname{Tr}B_-^\gamma\,.$$

For the proof of this proposition we need several preliminary results. The first one is an extension of Corollary 2.28. We refer to the discussion before that corollary for the precise definition of the operator T^*AT.

Lemma 2.41. *If A is self-adjoint and lower semi-bounded and if T is bounded, then for any $\gamma>0$,*

$$\operatorname{Tr}(T^*AT)_-^\gamma \leq \|T\|^{2\gamma}\operatorname{Tr}A_-^\gamma\,.$$

Proof. According to Lemma 2.36 we have

$$\operatorname{Tr}(T^*AT)_-^\gamma = \gamma \int_0^\infty N(-\kappa, T^*AT)\kappa^{\gamma-1}\,d\kappa\,.$$

For any $\kappa > 0$, we have $T^*AT + \kappa \geq T^*(A + \|T\|^{-2}\kappa)T$ and therefore

$$N(-\kappa, T^*AT) = N(0, T^*AT + \kappa) \leq N(0, T^*(A + \|T\|^{-2}\kappa)T)\,.$$

We now apply Corollary 2.28, which implies that

$$N(0, T^*(A + \|T\|^{-2}\kappa)T) \leq N(0, A + \|T\|^{-2}\kappa) = N(-\|T\|^{-2}\kappa, A)\,,$$

and therefore

$$\operatorname{Tr}(T^*AT)_-^\gamma \leq \gamma \int_0^\infty N(-\|T\|^{-2}\kappa, A)\kappa^{\gamma-1}\,d\kappa = \|T\|^{2\gamma}\operatorname{Tr}(A)_-^\gamma\,,$$

as claimed. □

The next result bounds the change in eigenvalues in an interval under a rank one perturbation.

Lemma 2.42. *Let A be a self-adjoint operator and let B be a rank-one operator. Then for any bounded, open interval I,*

$$\dim P_I(A)\mathscr{H} - 1 \leq \dim P_I(A+B)\mathscr{H} \leq \dim P_I(A)\mathscr{H} + 1\,,$$

where $P_I(A)$ and $P_I(A+B)$ denote the spectral projections corresponding to A and $A+B$, respectively. In particular, if A is lower semibounded and if $B \leq 0$, then

$$\lambda_1(A+B) \leq \lambda_1(A) \leq \lambda_2(A+B) \leq \lambda_2(A) \leq \ldots\,,$$

where $\lambda_n(A)$ and $\lambda_n(A+B)$ denote the eigenvalues below $\inf \sigma_{\mathrm{ess}}(A) = \inf \sigma_{\mathrm{ess}}(A+B)$ *in increasing order, counting multiplicities of A and $A+B$, respectively.*

Proof. Since $A = (A+B) - B$ with $-B$ rank-one, the left inequality follows once the right inequality is proved. To prove the latter, we argue by contradiction and assume that $\dim P_I(A+B)\mathscr{H} > \dim P_I(A)\mathscr{H} + 1$. Since the space $\operatorname{ran} P_I(A+B) \cap \ker B$ has dimension $\geq \dim P_I(A+B)\mathscr{H} - 1$, which is larger than the dimension of the space $\operatorname{ran} P_I(A)$, there is an $0 \neq x \in \operatorname{ran} P_I(A+B) \cap \ker B \cap (\operatorname{ran} P_I(A))^\perp$. Let μ denote the midpoint of the interval I. Then, since $x \in \operatorname{ran} P_I(A+B)$,

$$\|(A+B-\mu)x\|^2 = \int_I (\lambda - \mu)^2\,d\,(P_\lambda(A+B)x, x) < \frac{|I|^2}{4}\|x\|^2\,,$$

and, since $x \in (\operatorname{ran} P_I(A))^\perp$,

$$\|(A-\mu)x\|^2 = \int_{\mathbb{R}\setminus I} (\lambda - \mu)^2\,d\,(P_\lambda(A)x, x) \geq \frac{|I|^2}{4}\|x\|^2\,,$$

This is a contradiction, since $x \in \ker B$ implies that $(A+B-\mu)x = (A-\mu)x$.

We now prove the interlacing property of the eigenvalues under the assumption that $B \le 0$. We note that the identity $\inf \sigma_{\mathrm{ess}}(A) = \inf \sigma_{\mathrm{ess}}(A+B)$ follows from Weyl's theorem (Theorem 2.10). From the variational principle we obtain immediately that $\lambda_n(A+B) \le \lambda_n(A)$ for all n. We show the remaining inequality $\lambda_n(A) \le \lambda_{n+1}(A+B)$ via contradiction. Suppose that $\lambda_{n+1}(A+B) < \lambda_n(A)$ for some n. Then with $I = (\lambda_1(A+B)-1, \lambda_n(A))$,

$$n+1 \le \dim P_I(A+B)\mathscr{H}$$

and

$$\dim P_I(A)\mathscr{H} \le n-1\,.$$

This contradicts the inequality shown in the first part of the proof. □

The final ingredient in the proof of Proposition 2.40 is the following symmetrization inequality.

Lemma 2.43. *Let* $0 < \gamma < 1$. *Then for any set* $E \subset [0,\infty)$ *of finite measure*

$$\int_E t^{\gamma-1}\,dt \le \int_0^{|E|} t^{\gamma-1}\,dt\,.$$

Proof. Since

$$t^{\gamma-1} = (1-\gamma)\int_t^\infty s^{\gamma-2}\,ds = (1-\gamma)\int_0^\infty s^{\gamma-2}\chi_{(0,s)}(t)\,ds\,,$$

we have

$$\begin{aligned}\int_E t^{\gamma-1}\,dt &= (1-\gamma)\int_0^\infty \left(\int_0^\infty \chi_E(t)\chi_{(0,s)}(t)\,dt\right) s^{\gamma-2}\,ds \\ &= (1-\gamma)\int_0^\infty |E\cap(0,s)| s^{\gamma-2}\,ds\,.\end{aligned}$$

Obviously, $|E\cap(0,s)| \le \min\{|E|,s\}$. Note that equality holds here if $E=(0,|E|)$. Inserting this into the above identity, we obtain

$$\begin{aligned}\int_E t^{\gamma-1}\,dt &\le (1-\gamma)\left(\int_0^{|E|} s^{\gamma-1}\,ds + |E|\int_{|E|}^\infty s^{\gamma-2}\,ds\right) \\ &= (1-\gamma)\left(\frac{|E|^\gamma}{\gamma} + \frac{|E|^\gamma}{1-\gamma}\right) = \frac{|E|^\gamma}{\gamma} = \int_0^{|E|} t^{\gamma-1}\,dt\,,\end{aligned}$$

as claimed. □

Proof (Proof of Proposition 2.40). We begin with the case where B_- is rank-one and we denote its negative eigenvalue by $-\beta$. Moreover, we denote the negative eigenvalues of A and $A+B$ by $-\alpha_1 \le -\alpha_2 \le \ldots$ and $-\lambda_1 \le -\lambda_2 \le \ldots$. According to Lemma 2.42, we have

$$-\lambda_n \leq -\alpha_n \leq -\lambda_{n+1} \leq -\alpha_{n+1} \leq \dots \qquad \text{for all } n\,. \tag{2.36}$$

Then

$$\begin{aligned}\operatorname{Tr}(A+B)_-^\gamma &= \sum_n \lambda_n^\gamma = \gamma \sum_n \int_0^{\lambda_n} t^{\gamma-1}\,dt = \gamma \sum_n \int_0^{\alpha_n} t^{\gamma-1}\,dt + \gamma \sum_n \int_{\alpha_n}^{\lambda_n} t^{\gamma-1}\,dt \\ &= \operatorname{Tr} A_-^\gamma + \gamma \sum_n \int_{I_n} t^{\gamma-1}\,dt\,,\end{aligned}$$

where $I_n = (\alpha_n, \lambda_n)$. We note that by (2.36) the intervals I_n are all disjoint and therefore

$$\sum_n \int_{I_n} t^{\gamma-1}\,dt = \int_E t^{\gamma-1}\,dt$$

with $E = \bigcup_n I_n$. Moreover,

$$|E| = \sum_n |I_n| = \sum_n \lambda_n - \sum_n \alpha_n = \operatorname{Tr}(A+B)_- - \operatorname{Tr} A_- \leq \operatorname{Tr} B_- = \beta\,,$$

where the inequality comes from Corollary 2.35. Therefore, Lemma 2.43 implies that

$$\gamma \sum_n \int_{I_n} t^{\gamma-1}\,dt \leq \gamma \int_0^{|E|} t^{\gamma-1}\,dt \leq \gamma \int_0^{\beta} t^{\gamma-1}\,dt = \beta^\gamma = \operatorname{Tr} B_-^\gamma\,.$$

This proves the claimed inequality in the case where B_- is rank-one.

The case where B_- is of finite rank follows by iterating the inequality in the rank-one case.

We finally deal with the case of an arbitrary operator B with $\operatorname{Tr} B_-^\gamma < \infty$. For $\kappa > 0$ we denote $P_\kappa = \chi_{(-\infty,-\kappa)}(A+B)$. According to Lemma 2.41 we have

$$\operatorname{Tr}(P_\kappa A P_\kappa)_-^\gamma \leq \operatorname{Tr} A_-^\gamma < \infty \qquad \text{and} \qquad \operatorname{Tr}(P_\kappa B P_\kappa)_-^\gamma \leq \operatorname{Tr} B_-^\gamma < \infty\,. \tag{2.37}$$

We know from the non-sharp bound of Proposition 2.37 that under the assumption $\operatorname{Tr} A_-^\gamma + \operatorname{Tr} B_-^\gamma < \infty$, we also have $\operatorname{Tr}(A+B)_-^\gamma < \infty$ and therefore, in particular, P_κ has finite rank. Thus, $P_\kappa B P_\kappa$ has finite rank as well and, by what we have shown so far, we infer that $\operatorname{Tr}(P_\kappa A P_\kappa + P_\kappa B P_\kappa)_-^\gamma < \infty$ and

$$\operatorname{Tr}(P_\kappa A P_\kappa + P_\kappa B P_\kappa)_-^\gamma \leq \operatorname{Tr}(P_\kappa A P_\kappa)_-^\gamma + \operatorname{Tr}(P_\kappa B P_\kappa)_-^\gamma \leq \operatorname{Tr} A_-^\gamma + \operatorname{Tr} B_-^\gamma\,.$$

On the other hand,

$$\operatorname{Tr}(P_\kappa A P_\kappa + P_\kappa B P_\kappa)_-^\gamma = \operatorname{Tr}(P_\kappa (A+B) P_\kappa)_-^\gamma = \sum_{\lambda_n(A+B) < -\kappa} |\lambda_n(A+B)|^\gamma$$

and by monotone convergence, this converges to $\sum_n |\lambda_n(A+B)|^\gamma = \operatorname{Tr}(A+B)_-^\gamma$ as $\kappa \to \infty$. This proves the claimed inequality. □

2.2.6 Perturbations of quadratic forms

In the remainder of this subsection we take on a perturbation theoretic point of view. That is, there will be a quadratic form a which is lower semibounded and closed and corresponds to an operator A, and we will study self-adjointness of $A+B$, where B is, in a sense to be made precise, small with respect to A. We formulate this smallness in the sense of quadratic forms.

The following simple lemma is sometimes useful to verify that a perturbation of a lower semibounded, closed quadratic form is also lower semibounded and closed.

Lemma 2.44. *Assume that a is a lower semibounded, closed quadratic form with domain $d[a]$ and assume that b is a real quadratic form on $d[a]$ such that for some $\theta \in [0,1)$ and some $C \in \mathbb{R}$*

$$|b[u]| \leq \theta a[u] + C\|u\|^2 \qquad \text{for all } u \in d[a]\,. \tag{2.38}$$

Then the quadratic form $a+b$ with domain $d[a]$ is lower semibounded and closed.

Proof. By assumption, we have the inequalities

$$(1-\theta)a[u] - C\|u\|^2 \leq a[u] + b[u] \leq (1+\theta)a[u] + C\|u\|^2\,.$$

The inequality on the left shows the lower semiboundedness, and the proof of closedness is a simple exercise. □

As a consequence of Lemma 2.44 and Theorem 2.15, the quadratic form $a+b$ generates a self-adjoint operator with domain $d[a]$. Often we shall denote this operator by $A+B$, but we emphasize that this is an abuse of notation since, in general, there need not be a well-defined, self-adjoint operator B given by the difference of $A+B$ and A.

Our next goal is to derive a formula for the resolvent difference $(A+B-z)^{-1} - (A-z)^{-1}$. When the real quadratic form b corresponds to a bounded self-adjoint operator B, then it is well known that

$$(A+B-z)^{-1} - (A-z)^{-1} = -(A-z)^{-1}B(A+B-z)^{-1}\,.$$

This can be verified by applying $A-z$ from the left and $A+B-z$ from the right.

To motivate the formula that we are looking for in the general case, we write the left side as

$$-\left[(A-z)^{-1}(A+m)^{1/2}\right]\left[(A+m)^{-1/2}B(A+m)^{-1/2}\right]\left[(A+m)^{1/2}(A+B-z)^{-1}\right]$$

for some $m > -m_a$. As we will show, the analogue of each of the three factors in square brackets is well-defined under assumption (2.38). This is clear for the first factor. The next lemma essentially deals with the third factor.

Lemma 2.45. *Assume* (2.38). *Then for any $M > C/\theta$ the operator $(A+M)^{1/2}(A+B+M)^{-1/2}$ is bounded with*

$$\left\| (A+M)^{1/2}(A+B+M)^{-1/2} \right\| \leq \frac{1}{\sqrt{1-\theta}}\,.$$

Proof. Estimating the left side of (2.38) by zero, we see that

$$a[u] + \frac{C}{\theta}\|u\|^2 \geq 0\,,$$

that is, $C/\theta \geq -m_a$. According to (2.38) for any $M > C/\theta$ we have

$$a[u] + b[u] + M\|u\|^2 \geq (1-\theta)\left(a[u] + \frac{M-C}{1-\theta}\|u\|^2\right) \geq (1-\theta)\left(a[u] + M\|u\|^2\right)\,.$$

Since for $M > C/\theta$ the right side is positive definite, so is the left side, and the operator $(A+B+M)^{-1/2}$ exists and is bounded. Thus, we can set $u = (A+B+M)^{-1/2}v$, and the previous inequality becomes

$$\begin{aligned}\|v\|^2 &\geq (1-\theta)\left(a[(A+B+M)^{-1/2}v] + M\|(A+B+M)^{-1/2}v\|^2\right)\\ &= (1-\theta)\|(A+m)^{1/2}(A+B+M)^{-1/2}v\|^2\,.\end{aligned}$$

This proves that $(A+M)^{1/2}(A+B+M)^{-1/2}$ is bounded with the claimed bound on the norm. □

It follows from (2.38) that for any $m > -m_a$ there is a constant C' such that

$$|b[u]| \leq C'\left(a[u] + m\|u\|^2\right) \qquad \text{for all } u \in d[a]\,. \tag{2.39}$$

Let us fix an $m > -m_a$ and a corresponding norm $\left(a[u] + m\|u\|^2\right)^{1/2}$ on $d[a]$. By the Riesz representation theorem, the form b generates a bounded operator $\mathscr{B}_a$ on $d[a]$ in the sense that

$$b[u,v] = a[\mathscr{B}_a u, v] + m[\mathscr{B}_a u, v] \qquad \text{for all } u, v \in d[a]\,. \tag{2.40}$$

Clearly, the operator $\mathscr{U} : \mathscr{H} \to d[a]$, $f \mapsto (A+m)^{-1/2}f$ is unitary. Thus,

$$\hat{\mathscr{B}}_a = \mathscr{U}^* \mathscr{B}_a \mathscr{U}$$

is a bounded operator on $\mathscr{H}$. Note that in the case where b comes from a bounded operator B in $\mathscr{H}$, we have

$$\hat{\mathscr{B}}_a = (A+m)^{-1/2} B (A+m)^{-1/2}\,.$$

This operator appears in the formula for the resolvent difference.

Proposition 2.46. *Let a be a lower semibounded quadratic form and let b be a real quadratic form satisfying* (2.38) *for some* $\theta \in [0,1)$ *and* $C \in \mathbb{R}$. *Then the resolvents of the operators A and* $A+B$ *associated to a and* $a+b$ *are related by*

$$\begin{aligned}(A+B-z)^{-1}-(A-z)^{-1} &= \\ &= -\left[(A-z)^{-1}(A+m)^{1/2}\right]\hat{\mathscr{B}}_a\left[(A+m)^{1/2}(A+B-z)^{-1}\right]\end{aligned}$$

for any $z\in\rho(A)\cap\rho(A+B)$.

We emphasize that all three factors on the right side are bounded operators. For $\hat{\mathscr{B}}_a$ this was discuss before the proposition. For the first factor this is clear from the spectral theorem, and for the third factor it follows from

$$\begin{aligned}&(A+m)^{1/2}(A+B-z)^{-1}\\ &\quad=\left[(A+m)^{1/2}(A+M)^{-1/2}\right]\left[(A+M)^{1/2}(A+B+M)^{-1}\right]\\ &\quad\quad\times\left[(A+B+M)(A+B-z)^{-1}\right]\end{aligned}$$

together with Lemma 2.45 with $M>C/\theta$ and the spectral theorem.

Proof. We have to show that for any $f,g\in\mathscr{H}$,

$$\begin{aligned}&\left(\left((A+B-z)^{-1}-(A-z)^{-1}\right)f,g\right)\\ &\quad=-\left(\left[(A-z)^{-1}(A+m)^{1/2}\right]\hat{\mathscr{B}}_a\left[(A+m)^{1/2}(A+B-z)^{-1}\right]f,g\right).\end{aligned}$$

Since $z\in\rho(A)\cap\rho(A+B)$ we can define

$$u=(A+B-z)^{-1}f\,,\qquad v=(A-\bar{z})^{-1}g\,.$$

Then $u\in\operatorname{dom}(A+B)$ and $v\in\operatorname{dom}(A)$ and, in particular, $u,v\in d[a]$. Thus, by the definition of an operator associated to a quadratic form, we have for the left side above

$$\begin{aligned}\left(\left((A+B-z)^{-1}-(A-z)^{-1}\right)f,g\right)&=(u,(A-\bar{z})v)-((A+B-z)u,v)\\ &=a[u,v]-(a[u,v]+b[u,v])=-b[u,v]\,.\end{aligned}$$

Let us discuss the right side above. We have

$$\begin{aligned}&\left(\left[(A-z)^{-1}(A+m)^{1/2}\right]\hat{\mathscr{B}}_a\left[(A+m)^{1/2}(A+B-z)^{-1}\right]f,g\right)\\ &\quad=\left(\hat{\mathscr{B}}_a(A+m)^{1/2}u,(A+m)^{1/2}v\right)\\ &\quad=\left(\mathscr{U}^*\mathscr{B}_a\mathscr{U}\,(A+m)^{1/2}u,(A+m)^{1/2}v\right)\\ &\quad=(\mathscr{U}^*\mathscr{B}_a u,\mathscr{U}^*v)\\ &\quad=a[\mathscr{B}_a u,v]+m(\mathscr{B}_a u,v)\\ &\quad=b[u,v]\,.\end{aligned}$$

Thus both sides of the identity coincide, which proves the proposition. □

We say that the quadratic form b is *compact with respect to a* if for some (and hence any) $m > -m_a$ there is a constant C' such that (2.39) holds and if the operator $\mathscr{B}_a$ is compact on $d[a]$. (This is clearly equivalent to $\hat{\mathscr{B}}_a$ being compact in $\mathscr{H}$.)

Lemma 2.47. *Let a be a lower semibounded, closed form and assume that b is relatively compact with respect to a. Then for any $\theta > 0$ there is a C (depending on θ) such that* (2.38) *holds.*

Proof. We argue by contradiction. If the assertion of the lemma is false, then there is a sequence $(u_n) \subset d[a]$ with $a[u_n] + m\|u_n\|^2 = 1$ (for some $m > -m_a$) such that

$$|b[u_n]| \geq \theta + n\|u_n\|^2 \tag{2.41}$$

Weak compactness of the unit ball in $d[a]$ allows us extract a subsequence (u_{n_m}) which converges weakly in $d[a]$ to some $u \in d[a]$. Since b is compact with respect to a, we have $\mathscr{B}_a u_{n_m} \to \mathscr{B}_a u$ strongly in $d[a]$. Thus, $b[u_{n_m}] \to b[u]$. Thus, (2.41) implies that $n_m\|u_{n_m}\|^2$ is bounded, which means that $u_{n_m} \to 0$ in $\mathscr{H}$. This implies that $u = 0$, and then (2.41) leads to a contradiction since $\theta > 0$. □

We finish this subsection with a quadratic form version of Weyl's theorem (Theorem 2.10).

Theorem 2.48. *Let a be a lower semibounded, closed quadratic form and let b be a real quadratic form which is compact with respect to a. Then the operators A and $A + B$ associated to a and $a + b$ satisfy $\sigma_{\mathrm{ess}}(A) = \sigma_{\mathrm{ess}}(A + B)$.*

Proof. We want to apply Theorem 2.10 and therefore have to show that $(A + B - z)^{-1} - (A - z)^{-1}$ is compact. This follows immediately from the resolvent identity in Proposition 2.46, since the middle factor $\hat{\mathscr{B}}_a$ is compact by assumption and the outer factors are bounded as discussed after the theorem. This completes the proof. □

2.2.7 The Birman–Schwinger principle

In this subsection we assume that a is a non-negative closed quadratic form in a Hilbert space $\mathscr{H}$ with domain $d[a]$. In addition, we assume that

$$a[u] > 0 \qquad \text{for all } 0 \neq u \in d[a]\,. \tag{2.42}$$

This condition implies that $\sqrt{a[\cdot]}$ is a norm. We define $\mathscr{H}_a$ to be the completion of $d[a]$ with respect to the norm $\sqrt{a[\cdot]}$ and we denote by $\hat{a}$ the extension of a by continuity to $\mathscr{H}_a$.

If a is positive definite (that is, if the greatest lower bound $m_a > 0$), then $\mathscr{H}_a$ coincides with $d[a]$ (with equivalent norms). In general, this need not be the case and $\mathscr{H}_a$ may not be a subset of $\mathscr{H}$.

For practical purposes it is useful to note that if $\mathscr{F} \subset d[a]$ is dense in $d[a]$ (with respect to the norm $\sqrt{a[\cdot]+m\|\cdot\|^2}$ for some $m > -m_a$), then $\mathscr{H}_a$ coincides with the completion of $\mathscr{F}$ with respect to $\sqrt{a[\cdot]}$.

Let b be a real-valued quadratic form satisfying $d[b] \supset d[a]$ and

$$|b[u]| \leq Ma[u] \qquad \text{for all } u \in d[a]. \tag{2.43}$$

This implies that b can be extended by continuity to a quadratic form $\hat{b}$ on $\mathscr{H}_a$. This extended quadratic form defines a bounded, self-adjoint operator $\mathscr{B}_a$ on $\mathscr{H}_a$. (We emphasize that this operator is related, but different from the operator $\mathscr{B}_a$ appearing in (2.40). Indeed, the operator in (2.40) is defined on $d[a]$ with norm $\sqrt{a[\cdot]+m\|\cdot\|^2}$ where $m > -m_a$. Now, we allow for $m = -m_a$ which is why we have to introduce the larger space $\mathscr{H}_a$.)

The following result is called the *Birman–Schwinger principle*.

Theorem 2.49. *In addition to assumptions* (2.42) *and* (2.43) *assume that the quadratic form $a - \alpha b$ is lower semi-bounded and closed in $\mathscr{H}$ and denote by $A - \alpha B$ the corresponding self-adjoint operator in $\mathscr{H}$. Then*

$$\dim P_{(-\infty,0)}(A-\alpha B)\mathscr{H} = \dim P_{(\alpha^{-1},\infty)}(\mathscr{B}_a)\mathscr{H}_a\,, \tag{2.44}$$

where $P_{(-\infty,0)}(A-\alpha B)$ and $P_{(\alpha^{-1},\infty)}(\mathscr{B}_a)$ denote the spectral measures of $A - \alpha B$ and of $\mathscr{B}_a$, respectively.

Proof. If follows from Glazman's lemma (Theorem 2.23) applied to the operator $-\mathscr{B}_a$ in the Hilbert space $\mathscr{H}_a$ that for any $\lambda \in \mathbb{R}$

$$\dim P_{(\lambda,\infty)}(\mathscr{B}_a)\mathscr{H}_a = \sup\left\{\dim F :\ F \subset \mathscr{F} \text{ and } b[u] > \lambda a[u] \text{ for all } 0 \neq u \in F\right\},$$

where $\mathscr{F}$ is an arbitrary dense subspace of $\mathscr{H}_a$ (for instance, $d[a]$). On the other hand, applying Glazman's lemma (Theorem 2.23) to the operator $A - \alpha B$ in the Hilbert space $\mathscr{H}$, we find that

$$\begin{aligned}&\dim P_{(-\infty,0)}(A-\alpha B)\mathscr{H}\\ &\qquad = \sup\left\{\dim F :\ F \subset \mathscr{F} \text{ and } b[u] > \alpha^{-1}a[u] \text{ for all } 0 \neq u \in F\right\},\end{aligned}$$

where $\mathscr{F}$ is an arbitrary dense subspace of $d[a]$. Choosing $\lambda = \alpha^{-1}$ we obtain the claimed inequality. □

The reason why Theorem 2.49 is useful is that it relates the number of negative eigenvalues of a lower semi-bounded operator to the number of eigenvalues of a bounded (and typically compact) operator.

Conditions which guarantee that $a - \alpha b$ is lower semi-bounded and closed in $\mathscr{H}$ are given in Lemma 2.44 and Lemma 2.47.

We now state a second version of Theorem 2.49 where assumption (2.42) is replaced by the stronger assumption that $m_a > 0$, that is, there is an $\varepsilon > 0$ such that

$$a[u] \geq \varepsilon\|u\|^2 \qquad \text{for all } u \in d[a]. \tag{2.45}$$

Theorem 2.50. *In addition to assumptions* (2.45) *and* (2.43) *assume that the quadratic form $a-\alpha b$ is lower semi-bounded and closed in $\mathscr{H}$ and denote by $A-\alpha B$ the corresponding self-adjoint operator in $\mathscr{H}$. Then, in addition to* (2.44)*,*

$$\dim P_{(-\infty,0]}(A-\alpha B)\mathscr{H} = \dim P_{[\alpha^{-1},\infty)}(\mathscr{B}_a)\mathscr{H}_a \tag{2.46}$$

and

$$\dim\ker(A-\alpha B) = \dim\ker(\mathscr{B}_a-\alpha^{-1})\,. \tag{2.47}$$

Proof. The proof of (2.46) is the same as that of (2.44), except that one uses the second identity in Glazman's lemma (Theorem 2.23). This is possible since $d[a]$ is not only dense in $\mathscr{H}_a$, but actually equals $\mathscr{H}_a$. Identity (2.47) follows by subtracting (2.44) from (2.46).

For applications it is useful to reformulate the previous two theorems in terms of operators acting in the original Hilbert space only. We note that assumption (2.42) guarantees that $A^{-1/2}$ is a densely defined operator in $\mathscr{H}$.

Theorem 2.51. *In addition to assumptions* (2.42) *and* (2.43) *assume that the quadratic form $a-\alpha b$ is lower semi-bounded and closed in $\mathscr{H}$ and denote by $A-\alpha B$ the corresponding self-adjoint operator in $\mathscr{H}$. Then the quadratic form*

$$b[A^{-1/2}f, A^{-1/2}f]\,,\ f\in\operatorname{dom}A^{-1/2}\,,$$

is closable and bounded. Denoting by corresponding self-adjoint operator in $\mathscr{H}$ by $A^{-1/2}BA^{-1/2}$, we have

$$\dim P_{(-\infty,0)}(A-\alpha B)\mathscr{H} = \dim P_{(\alpha^{-1},\infty)}(A^{-1/2}BA^{-1/2})\mathscr{H}\,, \tag{2.48}$$

where $P_{(-\infty,0)}(A-\alpha B)$ and $P_{(\alpha^{-1},\infty)}(A^{-1/2}BA^{-1/2})$ denote the spectral measures of $A-\alpha B$ and of $A^{-1/2}BA^{-1/2}$, respectively.

Proof. We shall show that the operator $A^{-1/2}BA^{-1/2}$ in $\mathscr{H}$ is unitarily equivalent to the operator $\mathscr{B}_a$ in $\mathscr{H}_a$. Therefore the theorem follows from Theorem 2.49.

In order to prove the claimed unitary equivalence we consider the operator $A^{-1/2}$, defined on $\operatorname{dom}A^{-1/2}$, as a mapping from a dense subset of $\mathscr{H}$ into $d[a]=\operatorname{dom}A^{1/2}$ equipped with the norm $\sqrt{a[\cdot]}$. This mapping is clearly isometric, since

$$a[A^{-1/2}f] = \|f\|^2 \qquad \text{for all } f\in\operatorname{dom}A^{-1/2}\,.$$

Moreover, $\operatorname{ran}A^{-1/2}=d[a]$ by the spectral theorem. Since $\mathscr{H}_a$ is the completion of $d[a]$ with respect to $\sqrt{a[\cdot]}$, this implies that the above mapping extends to a unitary operator $U:\mathscr{H}\to\mathscr{H}_a$.

By definition of the operator $\mathscr{B}_a$ we have

$$\hat{a}[\mathscr{B}_a u,u] = \hat{b}[u] \qquad \text{for all } u\in\mathscr{H}_a\,.$$

We apply this identity to $u = Uf$ with $f \in \operatorname{dom} A^{-1/2}$. Then, since $Uf = A^{-1/2} f \in d[a] \subset d[b]$ for $f \in \operatorname{dom} A^{-1/2}$, we have $\hat{b}[u] = b[A^{-1/2} f]$ and therefore

$$\hat{a}[\mathscr{B}_a Uf, Uf] = b[A^{-1/2} f] \qquad \text{for all } f \in \operatorname{dom} A^{-1/2} .$$

Since $U : \mathscr{H} \to \mathscr{H}_a$ is unitary, the previous identity implies that

$$(U^* \mathscr{B}_a Uf, f) = b[A^{-1/2} f] \qquad \text{for all } f \in \operatorname{dom} A^{-1/2} .$$

By (2.43) the operator $\mathscr{B}_a$ is bounded on $\mathscr{H}_a$ and therefore $U^* \mathscr{B}_a U$ is bounded on $\mathscr{H}$. This implies that the quadratic form $b[A^{-1/2} f]$ defined on $\operatorname{dom} A^{-1/2}$ is closable and bounded and that the corresponding operator satisfies

$$U^* \mathscr{B}_a U = A^{-1/2} B A^{-1/2} .$$

This is the claimed unitary equivalence. □

Theorem 2.52. *Under the assumptions of Theorem 2.51, but with* (2.42) *replaced by* (2.45) *we have, in addition to* (2.48),

$$\dim P_{(-\infty,0]}(A - \alpha B)\mathscr{H} = \dim P_{[\alpha^{-1},\infty)}(A^{-1/2} B A^{-1/2})\mathscr{H} \tag{2.49}$$

and

$$\dim \ker(A - \alpha B) = \dim \ker(A^{-1/2} B A^{-1/2} - \alpha^{-1}) . \tag{2.50}$$

Proof. This theorem follows from Theorem 2.50 because of the unitary equivalence between $A^{-1/2} B A^{-1/2}$ and $\mathscr{B}_a$, which we have shown in the proof of Theorem 2.51. □

Remark 2.53. Under the assumption that the quadratic form a is positive definite in the sense of (2.45) the operator $A^{-1/2}$ is bounded and under the additional assumption that the quadratic form b is bounded in $\mathscr{H}$ it generates a bounded operator B in $\mathscr{H}$. Then the quadratic form $b[A^{-1/2} f]$, $f \in \operatorname{dom} A^{-1/2}$, is closed and the associated operator is simply the product the three bounded operators $A^{-1/2} B A^{-1/2}$. This motivates this notation in the general case.

We next discuss the special case where b is non-negative and where there is a closed operator Q such that $\operatorname{dom} Q \supset \operatorname{dom} A^{-1/2}$ and

$$b[A^{-1/2} f] = \|Qf\|^2 \qquad \text{for all } f \in \operatorname{dom} A^{-1/2} . \tag{2.51}$$

Under this assumption the operator $A^{-1/2} B A^{-1/2}$ in the previous two theorems is given by $Q^* Q$. Applying Remark 2.21 we therefore immediately obtain from the previous two theorems the following ones.

Theorem 2.54. *Under the assumptions of Theorem 2.51 and, in addition,* (2.51)*, we have*

$$\dim P_{(-\infty,0)}(A - \alpha B)\mathscr{H} = \dim P_{(\alpha^{-1},\infty)}(QQ^*)\mathscr{H} , \tag{2.52}$$

where $P_{(-\infty,0)}(A-\alpha B)$ and $P_{(\alpha^{-1},\infty)}(QQ^*)$ denote the spectral measures of $A-\alpha B$ and of QQ^*, respectively.

Theorem 2.55. *Under the assumptions of Theorem 2.51, but with* (2.42) *replaced by* (2.45) *we have, in addition to* (2.48),

$$\dim P_{(-\infty,0]}(A-\alpha B)\mathscr{H} = \dim P_{[\alpha^{-1},\infty)}(QQ^*)\mathscr{H} \tag{2.53}$$

and

$$\dim \ker(A-\alpha B) = \dim \ker(QQ^* - \alpha^{-1}) \,. \tag{2.54}$$

Remark 2.56. Under the assumption that the quadratic form a is positive definite in the sense of (2.45) the operator $A^{-1/2}$ is bounded and under the additional assumption that the quadratic form b is bounded and non-negative in $\mathscr{H}$ it generates a bounded operator B in $\mathscr{H}$. Then $Q = B^{1/2}A^{-1/2}$ and $Q^* = A^{-1/2}B^{1/2}$. Therefore the previous to theorems relate the spectral properties of $A-\alpha B$ to those of the operator $QQ^* = B^{1/2}A^{-1}B^{1/2}$, which is usually called the *Birman–Schwinger operator*. Equation (2.54) says that $A-\alpha B$ has eigenvalue 0 if and only if the Birman–Schwinger operator has eigenvalue α^{-1} and that the corresponding multiplicities coincide.

2.2.8 Example of the computation of a trace

We conclude this section with an example of an operator whose trace can be computed.

Lemma 2.57. *Let X be a separable, sigma-finite measure space and $Q \in L^2(X\times X)$. Then the operator Q in $L^2(X)$, defined by*

$$(Qu)(x) = \int_X Q(x,x')u(x')\,dx' \,,$$

satisfies

$$\operatorname{Tr} Q^*Q = \iint_{X\times X} |Q(x,x')|^2\,dx\,dx' \,.$$

Proof. For any $u, v \in L^2(X)$, we have by the Schwarz inequality

$$\begin{aligned}
|(Qu,v)| &= \left| \iint_{X\times X} Q(x,x')u(x')\overline{v(x)}\,dx\,dx' \right| \\
&\le \left(\iint_{X\times X} |Q(x,x')|^2\,dx\,dx' \right)^{1/2} \left(\iint_{X\times X} |u(x')\overline{v(x)}|^2\,dx\,dx' \right)^{1/2} \\
&= \|Q\|\|u\|\|v\| \,.
\end{aligned}$$

This implies that Q is a bounded operator in $L^2(X)$.

Let $(u_j)_{j=1}^N$ be an orthonormal system in $L^2(X)$. We extend it to complete orthonormal system (u_j) in $L^2(X)$. Then

$$\sum_{j=1}^N \|Qu_j\|^2 = \sum_{j=1}^N \sum_i \big|(Qu_j,u_i)\big|^2 = \sum_{j=1}^N \sum_i \big|(Q,u_i\otimes\overline{u_j})\big|^2$$

where the inner product on the right side is in $L^2(X\times X$ and where the function $u_i\otimes\overline{u_j}$ is defined by

$$(u_i\otimes\overline{u_j})\,(x,x') = u_i(x)\overline{u_j(x')}\,, \qquad \text{for all } (x,x')\in X\times X\,.$$

It is easy to see that $(u_i\otimes\overline{u_j})$ is an orthonormal basis in $L^2(X\times X)$ and therefore, by Bessel's inequality we conclude that

$$\sum_{j=1}^N \sum_i \big|(Q,u_i\otimes\overline{u_j})\big|^2 \le \|Q\|^2\,.$$

According to the variational principle in Corollary 2.33 we conclude that

$$\operatorname{Tr} Q^*Q \le \|Q\|^2\,.$$

Moreover, by taking the supremum over N in the above argument it is easy to see that, in fact, $\operatorname{Tr} Q^*Q = \|Q\|^2$. □

We need the next corollary only for $\mathscr{G} = \mathbb{C}^N$.

Corollary 2.58. *Let X be a separable, sigma-finite measure space and let $\mathscr{G}$ be a separable Hilbert space. Let Q be a measurable function on $X\times X$ taking values in the bounded operators on $\mathscr{G}$ such that*

$$\iint_{X\times X} \operatorname{Tr}_{\mathscr{G}}\big((Q(x,x'))^*Q(x,x')\big)\,dx\,dx' < \infty\,.$$

Then the operator Q in $L^2(X,\mathscr{G})$, defined by

$$(Qu)(x) = \int_X Q(x,x')u(x')\,dx'\,,$$

satisfies

$$\operatorname{Tr} Q^*Q = \iint_{X\times X} \operatorname{Tr}_{\mathscr{G}}\big((Q(x,x'))^*Q(x,x')\big)\,dx\,dx'\,.$$

Proof. We assume that $\dim\mathscr{G} = \infty$, the finite-dimensional case differs only by simple change in notation. We fix a basis $(e_n)_{n\in\mathbb{N}}$ in $\mathscr{G}$ and consider the map $(\mathscr{J}u)(x,n) := (u(x),e_n)$, where the inner product on the right side is taken in $\mathscr{G}$. Then $\mathscr{J}$ is a unitary operator from $L^2(X,\mathscr{G})$ to $L^2(X\times\mathbb{N},\mathbb{C})$. Moreover,

$$(\mathscr{J}\,Q\,\mathscr{J}^*v)(x,n) = \int_X \sum_{n'\in\mathbb{N}} \tilde{Q}(x,n,x',n')v(x',n')\,dx'$$

where

$$\tilde{Q}(x,n,x',n') = \big(Q(x,x')e_{n'},e_n\big)\,.$$

We observe that, by completeness of the (e_n),

$$\sum_{n,n'} \iint_{X\times X} |\tilde{Q}(x,n,x',n')|^2\,dx\,dx' = \sum_{n'} \iint_{X\times X} \|Q(x,,x')e_{n'}\|_{\mathscr{G}}^2\,dx\,dx'\,.$$

By Lemma 2.34 we infer that for almost every $(x,x') \in X\times X$,

$$\sum_{n'} \|Q(x,,x')e_{n'}\|_{\mathscr{G}}^2 = \mathrm{Tr}_{\mathscr{G}}\big((Q(x,x'))^* Q(x,x')\big)\,.$$

Thus, by assumption, we have $\tilde{Q} \in L^2((X\times\mathbb{N})\times(X\times\mathbb{N}))$ and therefore the previous lemma, applied with X replaced by $X\times\mathbb{N}$, implies that

$$\begin{aligned}\mathrm{Tr}\,Q^*Q = \mathrm{Tr}(\mathscr{J}\,Q\,\mathscr{J}^*)^*(\mathscr{J}\,Q\,\mathscr{J}^*) &= \sum_{n,n'} \iint_{X\times X} |\tilde{Q}(x,n,x',n')|^2\,dx\,dx' \\ &= \iint_{X\times X} \mathrm{Tr}_{\mathscr{G}}\big((Q(x,x'))^* Q(x,x')\big)\,dx\,dx'\,,\end{aligned}$$

as claimed. □