M1M1: Problem Sheet 7: Integration

1. Integrate the following functions of x:

$$(a)\ \frac{x+1}{x};\quad (b)\ \frac{x}{x+1};\quad (c)\ \frac{x+1}{x-1}.$$

2. Using partial fractions (or otherwise) find the integrals of

$$(a)\ \frac{1}{x(2-3x)};\quad (b)\ \frac{x}{x^2-1};\quad (c)\ \frac{x^2}{x^3-1};\quad (d)\ \frac{1}{x(x^2+1)}.$$

3. Use an appropriate substitution to integrate:

$$(a)\ \frac{e^x}{e^x+1};\quad (b)\ \sin^2 x\cos x;\quad (c)\ \frac{\sin x}{1+\cos x};\quad (d)\ \frac{1}{\sqrt{1-x^2}};$$

$$(e)\ \frac{1}{\sqrt{x^2-1}};\quad (f)\ \cos x\, e^{\sin x}.$$

4. Use 'integration by parts' to integrate:

$$(a)\ e^x\cos x;\quad (b)\ \log x;\quad (c)\ x^2\cos x;\quad (d)\ \cos^{-1} x.$$

5. Integrate the following functions (using any appropriate method):

$$(a)\ \sqrt{x^2-1};\quad (b)\ \frac{\sqrt{x}}{1+x};\quad (c)\ \operatorname{cosec} x;\quad (d)\ \frac{\tan^{-1}(x)}{1+x^2}.$$

Evaluate the following definite integrals:

$$(e)\ \int_0^{\pi/2}\frac{dx}{5+4\cos x};\quad (f)\ \int_0^{\pi/2}\frac{dx}{(\sin x+\cos x)^2};$$

$$(g)\ \int_0^1\frac{dx}{(1+x^2)^{3/2}};\quad (h)\ \int_1^2\frac{dx}{x^2+3x+1}.$$

6. Show that [NB **MAPLE** can't do this!]

$$\int_0^{\pi}\frac{x\,dx}{1+\cos^2 x}=\frac{\pi}{2}\int_0^{\pi}\frac{dy}{1+\cos^2 y}=\frac{\pi^2}{2^{3/2}}.$$

7. Show that

$$\int_0^{\pi} \frac{dx}{\alpha - \cos x} = \frac{\pi}{\sqrt{\alpha^2 - 1}}, \qquad \alpha > 1.$$

8. The integrals $I(t_0)$ and $J(t_0)$ are defined as

$$I(t_0) = \int_{-\infty}^{\infty} \frac{dt}{\cosh t + \cosh t_0}, \qquad t_0 > 0,$$

$$J(t_0) = \int_{-\infty}^{\infty} \frac{dt}{\cosh t + \cos t_0}, \qquad 0 < t_0 < \pi/2.$$

Using the substitution $p = e^t$, or otherwise, show that

$$I(t_0) = \frac{2t_0}{\sinh t_0}, \qquad J(t_0) = \frac{2t_0}{\sin t_0}.$$

9. Determine whether or not the integral

$$\int_0^{\infty} f(x)\, dx \qquad \text{exists in the following cases:}$$

(a) $f(x) = x^{\alpha}\sqrt{2 + \cos x}$ where α is a real constant.

(b) $f(x) = (x^2 - 1)^{-1/3}$.

(c) $f(x) = \dfrac{1}{(x+1)\log x}$.

(d) $f(x) = \dfrac{(x+1)\sin x}{x^{3/2}(x - \pi)}$.

10. If

$$F_n = \int_0^1 x^n e^x dx$$

show that

$$F_n = e - nF_{n-1}, \qquad n = 1, 2, ...$$

Hence, evaluate F_4.

11. If

$$I_n = \int_0^{\infty} x^n e^{-x^2} dx$$

where n is a positive integer, show that

$$I_n = \left(\frac{n-1}{2}\right) I_{n-2} \qquad \text{for } n > 1.$$

Hence, evaluate I_5.

12. Given that

$$u_n(x) = \int x^n \cos x \, dx, \qquad v_n(x) = \int x^n \sin x \, dx,$$

by performing a single (complex) integration, show that

$$\begin{aligned} u_n(x) &= x^n \sin x - n v_{n-1}(x) \\ v_n(x) &= -x^n \cos x + n u_{n-1}(x) \end{aligned}$$

Hence, evaluate

$$\int x^4 \sin x dx.$$

13. If

$$I_n = \int_0^{\pi/4} \tan^n x dx,$$

show that for $n > 1$

$$I_n = \frac{1}{n-1} - I_{n-2}.$$

Hence, evaluate I_5.

14. If

$$I_n = \int_0^1 x^n \sqrt{1+x} \; dx, \quad n = 0, 1, 2, \ldots$$

show, by first bounding the integrand, that

$$\frac{1}{n+1} < I_n < \frac{\sqrt{2}}{n+1}.$$

Derive the recurrence relation for $n > 0$

$$(3 + 2n) I_n = 2^{5/2} - 2n I_{n-1}.$$

From the two previous results, deduce that

$$I_n > \frac{\sqrt{2}}{n + 3/2},$$

and using the various inequalities you have found, prove that as $n \to \infty$,

$$n I_n \to \sqrt{2}.$$

Line, Area and Volume integrals

15. Find the length of the curve $y = \sqrt{1-x^2}$ between $x = 0$ and $x = 1$.

16. The *cycloid* curve we considered in lectures was given parametrically by $x = t - \sin t$, $y = 1 - \cos t$. Find the length of this curve between the points $(0, 0)$ and $(2\pi, 0)$.

17. The portion of the parabola $y = 1 - x^2$ in $y > 0$ is rotated about the x-axis. What is the volume of the shape so formed?

18. The curve $y = 1/x$ for $x > 1$ is rotated about the x-axis. Calculate, if possible, the volume and the surface area so produced. Do you find the result strange?

19. Consider the integral of $\exp(-x^2 - y^2)$ over the whole (x, y)-plane,

$$I = \iint_{\mathbb{R}^2} e^{-x^2-y^2}\, dA.$$

(a) Using polar coordinates, and performing the r and θ integrals separately, show that $I = \pi$.
(b) using Cartesian coordinates, show that, $I = J^2$ where

$$J = \int_{-\infty}^{\infty} e^{-t^2}\, dt, \qquad \text{and deduce that } J = \sqrt{\pi}.$$

20. Show that the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{is} \quad \pi ab.$$

Choosing a suitable parameterisation, show that its circumference, L, is

$$L = \int_0^{2\pi} \left(a^2 \sin^2\theta + b^2 \cos^2\theta\right)^{1/2} d\theta = C \int_0^{\pi} (1 - k\cos\phi)^{1/2} d\phi,$$

for some constants C and k. (N.B. This integral cannot be evaluated in terms of elementary functions. It is known as an *elliptic integral.*)

This is quite a long sheet. If you don't manage to do it all, try at least a representative mixture from questions 1-8, 9, 10-14 and 15-20.